Problem 24

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0; \qquad y_1(t) = t$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(t)$ is also a solution: $cy_1(t) = ct$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of t.

$$y(t) = c(t)t$$

Substitute this formula for $y(t)$ into the ODE.

$$t^2[c(t)t]'' + 2t[c(t)t]' - 2[c(t)t] = 0$$

Evaluate the derivatives using the product rule.

$$t^2[c'(t)t + c(t)]' + 2t[c'(t)t + c(t)] - 2[c(t)t] = 0$$

$$t^2[c''(t)t + c'(t) + c'(t)] + 2t[c'(t)t + c(t)] - 2[c(t)t] = 0$$

$$t^3c''(t) + 2t^2c'(t) + 2t^2c'(t) + \cancel{2tc(t)} - \cancel{2tc(t)} = 0$$

$$t^3c''(t) + 4t^2c'(t) = 0$$

Divide both sides by t^3.

$$c''(t) + \frac{4}{t}c'(t) = 0$$

This is a linear first-order ODE for $c'(t)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{4}{s}\,ds\right) = e^{4\ln t} = e^{\ln t^4} = t^4$$

Proceed with the multiplication.

$$t^4c''(t) + 4t^3c'(t) = 0$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}[t^4c'(t)] = 0$$

Integrate both sides with respect to t.

$$t^4c'(t) = C_1$$

Divide both sides by t^4.

$$c'(t) = \frac{C_1}{t^4}$$

Integrate both sides with respect to t once more.

$$c(t) = -\frac{C_1}{3t^3} + C_2$$

Therefore, using a new constant C_3 for $-C_1/3$, the general solution is

$$y(t) = \frac{C_3}{t^2} + C_2 t;$$

the second solution is $y_2(t) = 1/t^2$.