Problem 26

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \qquad y_1(t) = t$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(t)$ is also a solution: $cy_1(t) = ct$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of t.

$$y(t) = c(t)t$$

Substitute this formula for $y(t)$ into the ODE.

$$t^2[c(t)t]'' - t(t+2)[c(t)t]' + (t+2)[c(t)t] = 0$$

Evaluate the derivatives using the product rule.

$$t^2[c'(t)t + c(t)]' - t(t+2)[c'(t)t + c(t)] + (t+2)[c(t)t] = 0$$

$$t^2[c''(t)t + c'(t) + c'(t)] - t(t+2)[c'(t)t + c(t)] + (t+2)[c(t)t] = 0$$

$$t^3c''(t) + \cancel{2t^2c'(t)} - t^3c'(t) - \cancel{2t^2c'(t)} - \cancel{t^2c(t)} - \cancel{2tc(t)} + \cancel{t^2c(t)} + \cancel{2tc(t)} = 0$$

$$t^3c''(t) - t^3c'(t) = 0$$

Divide both sides by t^3.

$$c''(t) - c'(t) = 0$$

This is a linear first-order ODE for $c'(t)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t (-1)\,ds\right) = e^{-t}$$

Proceed with the multiplication.

$$e^{-t}c''(t) - e^{-t}c'(t) = 0$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}[e^{-t}c'(t)] = 0$$

Integrate both sides with respect to t.

$$e^{-t}c'(t) = C_1$$

Multiply both sides by e^t.

$$c'(t) = C_1e^t$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 e^t + C_2$$

Therefore, the general solution is

$$y(t) = C_1 t e^t + C_2 t;$$

the second solution is $y_2(t) = te^t$.