Problem 34

In each of Problems 33 through 36, use the method of Problem 32 to find a second independent solution of the given equation.

$$ty'' - y' + 4t^3 y = 0, \quad t > 0; \qquad y_1(t) = \sin(t^2)$$

Solution

Divide both sides of the ODE by t to put it in the proper form.

$$y'' - \frac{1}{t}y' + 4t^2 y = 0$$

Use the result of Problem 32 to determine a second solution.

$$\begin{aligned} y_2(t) &= y_1(t) \int^t \frac{1}{[y_1(r)]^2} \exp\left[-\int^r p(s)\,ds\right] dr \\ &= \sin(t^2) \int^t \frac{1}{[\sin(r^2)]^2} \exp\left(\int^r \frac{1}{s}\,ds\right) dr \\ &= \sin t^2 \int^t \frac{1}{\sin^2 r^2} \exp(\ln r)\,dr \\ &= \sin t^2 \int^t r \csc^2 r^2\,dr \end{aligned}$$

Make the substitution,

$$\begin{aligned} u &= r^2 \\ du &= 2r\,dr \quad \rightarrow \quad \frac{du}{2} = r\,dr, \end{aligned}$$

in the integral.

$$\begin{aligned} y_2(t) &= \sin t^2 \int^{t^2} \csc^2 u \, \frac{du}{2} \\ &= \frac{\sin t^2}{2} \int^{t^2} \csc^2 u\,du \\ &= \frac{\sin t^2}{2}(-\cot u)\bigg|^{t^2} \\ &= -\frac{\sin t^2}{2} \cot t^2 \\ &= -\frac{\cos t^2}{2} \end{aligned}$$

Because the ODE is homogeneous, any constant multiple of $y_2(t)$ is also a solution.