Problem 35

In each of Problems 33 through 36, use the method of Problem 32 to find a second independent solution of the given equation.

$$(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$$

Solution

Divide both sides of the ODE by $x-1$ to put it in the proper form.

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$$

Use the result of Problem 32 to determine a second solution.

$$\begin{aligned}
y_2(x) &= y_1(x)\int^x \frac{1}{[y_1(r)]^2}\exp\left[-\int^r p(s)\,ds\right]dr \\
&= e^x \int^x \frac{1}{(e^r)^2}\exp\left(\int^r \frac{s}{s-1}\,ds\right)dr \\
&= e^x \int^x \frac{1}{(e^r)^2}\exp\left[r + \ln(r-1)\right]dr \\
&= e^x \int^x \frac{1}{(e^r)^2} e^r e^{\ln(r-1)}\,dr \\
&= e^x \int^x e^{-r}(r-1)\,dr \\
&= e^x \left(\int^x re^{-r} - \int^x e^{-r}\,dr\right) \\
&= e^x \left(-xe^{-x} - \cancel{e^{-x}} + \cancel{e^{-x}}\right) \\
&= e^x(-xe^{-x}) \\
&= -x
\end{aligned}$$

Because the ODE is homogeneous, any constant multiple of $y_2(x)$ is also a solution.