Problem 36

In each of Problems 33 through 36, use the method of Problem 32 to find a second independent solution of the given equation.

$$x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$$

Solution

Divide both sides of the ODE by x^2 to put it in the proper form.

$$y'' + \frac{1}{x}y' + \left(x - \frac{0.25}{x}\right)y = 0$$

Use the result of Problem 32 to determine a second solution.

$$\begin{aligned}
y_2(x) &= y_1(x)\int^x \frac{1}{[y_1(r)]^2}\exp\left[-\int^r p(s)\,ds\right]dr \\
&= x^{-1/2}\sin x\int^x \frac{1}{(r^{-1/2}\sin r)^2}\exp\left(-\int^r \frac{1}{s}\,ds\right)dr \\
&= x^{-1/2}\sin x\int^x \frac{1}{r^{-1}\sin^2 r}\exp\left(-\ln r\right)dr \\
&= x^{-1/2}\sin x\int^x r\csc^2 r e^{\ln r^{-1}}\,dr \\
&= x^{-1/2}\sin x\int^x r\csc^2 r(r^{-1})\,dr \\
&= x^{-1/2}\sin x\int^x \csc^2 r\,dr \\
&= x^{-1/2}\sin x(-\cot x) \\
&= -x^{-1/2}\cos x
\end{aligned}$$

Because the ODE is homogeneous, any constant multiple of $y_2(x)$ is also a solution.