Problem 12

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' - 6y' + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 2$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 6(re^{rt}) + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = \{3\}$$

One solution to the ODE is $y = e^{3t}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{3t}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{3t}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$y'' - 6y' + 9y = 0 \quad \to \quad [c(t)e^{3t}]'' - 6[c(t)e^{3t}]' + 9[c(t)e^{3t}] = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^{3t} + 3c(t)e^{3t}]' - 6[c'(t)e^{3t} + 3c(t)e^{3t}] + 9[c(t)e^{3t}] = 0$$

$$[c''(t)e^{3t} + 3c'(t)e^{3t} + 3c'(t)e^{3t} + 9c(t)e^{3t}] - 6[c'(t)e^{3t} + 3c(t)e^{3t}] + 9[c(t)e^{3t}] = 0$$

$$c''(t)e^{3t} + \cancel{3c'(t)e^{3t}} + \cancel{3c'(t)e^{3t}} + \cancel{9c(t)e^{3t}} - \cancel{6c'(t)e^{3t}} - \cancel{18c(t)e^{3t}} + \cancel{9c(t)e^{3t}} = 0$$

$$c''(t)e^{3t} = 0$$

Divide both sides by e^{3t}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 te^{3t} + C_2 e^{3t}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{3t} + 3C_1 t e^{3t} + 3C_2 e^{3t}$$

Apply the initial conditions now to determine C_1 and C_2.

$$y(0) = C_2 = 0$$
$$y'(0) = C_1 + 3C_2 = 2$$

Solving this system of equations yields $C_1 = 2$ and $C_2 = 0$. Therefore,

$$y(t) = 2te^{3t}.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 2te^{3t} = \infty$$

Below is a plot of $y(t)$ versus t.