Problem 15

Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \leq t \leq 5$.

(b) Determine where the solution has the value zero.

(c) Determine the coordinates (t_0, y_0) of the minimum point.

(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2 e^{rt}) + 12(re^{rt}) + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$4r^2 + 12r + 9 = 0$$

$$(2r + 3)^2 = 0$$

$$r = \left\{-\frac{3}{2}\right\}$$

One solution to the ODE is $y = e^{-3t/2}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-3t/2}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-3t/2}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$4y'' + 12y' + 9y = 0 \quad \rightarrow \quad 4[c(t)e^{-3t/2}]'' + 12[c(t)e^{-3t/2}]' + 9[c(t)e^{-3t/2}] = 0$$

Evaluate the derivatives using the product rule.

$$4\left[c'(t)e^{-3t/2} - \frac{3}{2}c(t)e^{-3t/2}\right]' + 12\left[c'(t)e^{-3t/2} - \frac{3}{2}c(t)e^{-3t/2}\right] + 9[c(t)e^{-3t/2}] = 0$$

$$4\left[c''(t)e^{-3t/2} - \frac{3}{2}c'(t)e^{-3t/2} - \frac{3}{2}c'(t)e^{-3t/2} + \frac{9}{4}c(t)e^{-3t/2}\right] + 12\left[c'(t)e^{-3t/2} - \frac{3}{2}c(t)e^{-3t/2}\right] + 9[c(t)e^{-3t/2}] = 0$$

$$4c''(t)e^{-3t/2} - \cancel{6c'(t)e^{-3t/2}} - \cancel{6c'(t)e^{-3t/2}} + \cancel{9c(t)e^{-3t/2}} + \cancel{12c'(t)e^{-3t/2}} - \cancel{18c(t)e^{-3t/2}} + \cancel{9c(t)e^{-3t/2}} = 0$$

$$4c''(t)e^{-3t/2} = 0$$

Divide both sides by $4e^{-3t/2}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 t e^{-3t/2} + C_2 e^{-3t/2}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{-3t/2} - \frac{3}{2} C_1 t e^{-3t/2} - \frac{3}{2} C_2 e^{-3t/2}$$

Apply the initial conditions now to determine C_1 and C_2.

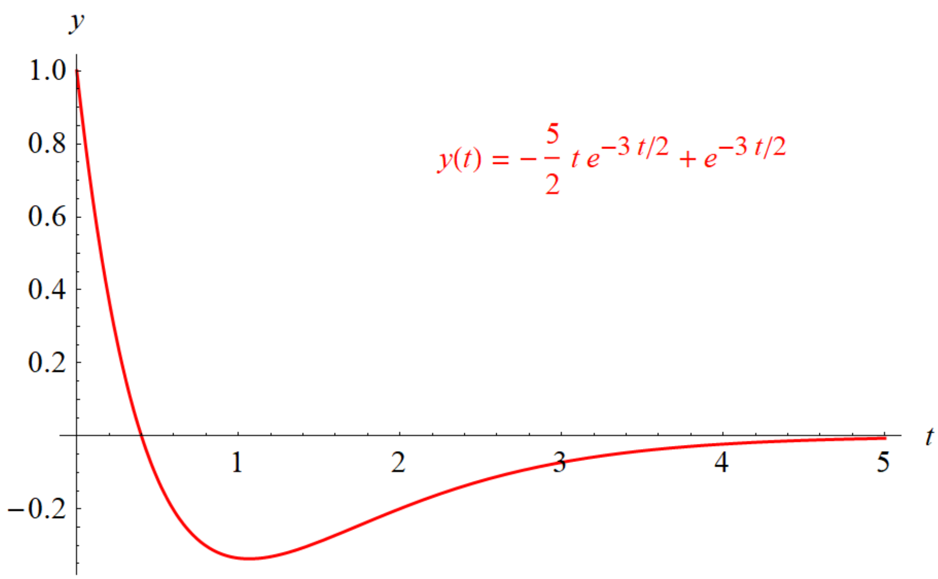
$$y(0) = C_2 = 1$$

$$y'(0) = C_1 - \frac{3}{2} C_2 = -4$$

Solving this system of equations yields $C_1 = -5/2$ and $C_2 = 1$. Therefore,

$$y(t) = -\frac{5}{2} t e^{-3t/2} + e^{-3t/2}.$$

Below is a plot of $y(t)$ versus t.



Find the zero by solving $y(t) = 0$ for t.

$$-\frac{5}{2} t e^{-3t/2} + e^{-3t/2} = 0$$

$$-\frac{5}{2} t + 1 = 0$$

$$t = \frac{2}{5} = 0.4$$

Differentiate the solution with respect to t to get $y'(t)$.

$$\begin{aligned} y'(t) &= -\frac{5}{2}e^{-3t/2} + \frac{15}{4}te^{-3t/2} - \frac{3}{2}e^{-3t/2} \\ &= \frac{15}{4}te^{-3t/2} - 4e^{-3t/2} \end{aligned}$$

Solve $y'(t) = 0$ for t to find the value of t at the minimum.

$$\frac{15}{4}te^{-3t/2} - 4e^{-3t/2} = 0$$

$$\frac{15}{4}t - 4 = 0$$

$$t = \frac{16}{15}$$

The value of y at this value of t is

$$y\left(\frac{16}{15}\right) = -\frac{5}{3e^{8/5}} \approx -0.336.$$

Therefore, the minimum is at

$$\left(\frac{16}{15}, -\frac{5}{3e^{8/5}}\right).$$

Return to the general solution and its derivative.

$$\begin{aligned} y(t) &= C_1te^{-3t/2} + C_2e^{-3t/2}. \\ y'(t) &= C_1e^{-3t/2} - \frac{3}{2}C_1te^{-3t/2} - \frac{3}{2}C_2e^{-3t/2} \end{aligned}$$

Apply the new initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(0) &= C_2 = 1 \\ y'(0) &= C_1 - \frac{3}{2}C_2 = b \end{aligned}$$

Solving this system of equations yields $C_1 = b + 3/2$ and $C_2 = 1$. Therefore,

$$\begin{aligned} y(t) &= \left(b + \frac{3}{2}\right)te^{-3t/2} + e^{-3t/2} \\ &= \left[\left(b + \frac{3}{2}\right)t + 1\right]e^{-3t/2}. \end{aligned}$$

For $y(t)$ to always be positive, the quantity in square brackets must always be positive. More specifically, the coefficient of t must be positive. The value of b that separates positive and negative solutions is

$$b + \frac{3}{2} = 0$$

$$b = -\frac{3}{2}.$$