Problem 16

Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b, and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - (re^{rt}) + 0.25(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - r + 0.25 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4(1)(0.25)}}{2} = \frac{1}{2}$$

$$r = \left\{ \frac{1}{2} \right\}$$

One solution to the ODE is $y = e^{t/2}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{t/2}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{t/2}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$y'' - y' + 0.25y = 0 \quad \rightarrow \quad [c(t)e^{t/2}]'' - [c(t)e^{t/2}]' + 0.25[c(t)e^{t/2}] = 0$$

Evaluate the derivatives using the product rule.

$$\left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right]' - \left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right] + 0.25[c(t)e^{t/2}] = 0$$

$$\left[c''(t)e^{t/2} + \frac{1}{2}c'(t)e^{t/2} + \frac{1}{2}c'(t)e^{t/2} + \frac{1}{4}c(t)e^{t/2}\right] - \left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right] + 0.25[c(t)e^{t/2}] = 0$$

$$c''(t)e^{t/2} + \cancel{\frac{1}{2}c'(t)e^{t/2}} + \cancel{\frac{1}{2}c'(t)e^{t/2}} + \cancel{\frac{1}{4}c(t)e^{t/2}} - \cancel{c'(t)e^{t/2}} - \cancel{\frac{1}{2}c(t)e^{t/2}} + \cancel{0.25c(t)e^{t/2}} = 0$$

$$c''(t)e^{t/2} = 0$$

Divide both sides by $e^{t/2}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 t e^{t/2} + C_2 e^{t/2}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{t/2} + \frac{1}{2} C_1 t e^{t/2} + \frac{1}{2} C_2 e^{t/2}$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(0) &= C_2 = 2 \\ y'(0) &= C_1 + \frac{1}{2} C_2 = b \end{aligned}$$

Solving this system of equations yields $C_1 = b - 1$ and $C_2 = 2$. Therefore,

$$\begin{aligned} y(t) &= (b-1)t e^{t/2} + 2e^{t/2} \\ &= [(b-1)t + 2]e^{t/2}. \end{aligned}$$

For $y(t)$ to always be positive, the quantity in square brackets must always be positive. More specifically, the coefficient of t must be positive. The value of b that separates positive and negative solutions is

$$b - 1 = 0$$

$$b = 1.$$