Problem 18

Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.

(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$9(r^2e^{rt}) + 12(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$9r^2 + 12r + 4 = 0$$
$$(3r+2)^2 = 0$$
$$r = \left\{-\frac{2}{3}\right\}$$

One solution to the ODE is $y = e^{-2t/3}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-2t/3}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-2t/3}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$9y'' + 12y' + 4y = 0 \quad \rightarrow \quad 9[c(t)e^{-2t/3}]'' + 12[c(t)e^{-2t/3}]' + 4[c(t)e^{-2t/3}] = 0$$

Evaluate the derivatives using the product rule.

$$9\left[c'(t)e^{-2t/3} - \frac{2}{3}c(t)e^{-2t/3}\right]' + 12\left[c'(t)e^{-2t/3} - \frac{2}{3}c(t)e^{-2t/3}\right] + 4[c(t)e^{-2t/3}] = 0$$

$$9\left[c''(t)e^{-2t/3} - \frac{2}{3}c'(t)e^{-2t/3} - \frac{2}{3}c'(t)e^{-2t/3} + \frac{4}{9}c(t)e^{-2t/3}\right] + 12\left[c'(t)e^{-2t/3} - \frac{2}{3}c(t)e^{-2t/3}\right] + 4[c(t)e^{-2t/3}] = 0$$

$$9c''(t)e^{-2t/3} - \cancel{6c'(t)e^{-2t/3}} - \cancel{6c'(t)e^{-2t/3}} + \cancel{4c(t)e^{-2t/3}} + \cancel{12c'(t)e^{-2t/3}} - \cancel{8c(t)e^{-2t/3}} + \cancel{4c(t)e^{-2t/3}} = 0$$
$$9c''(t)e^{-2t/3} = 0$$

Divide both sides by $9e^{-2t/3}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 t e^{-2t/3} + C_2 e^{-2t/3}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{-2t/3} - \frac{2}{3} C_1 t e^{-2t/3} - \frac{2}{3} C_2 e^{-2t/3}$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned} y(0) &= C_2 = a \\ y'(0) &= C_1 - \frac{2}{3} C_2 = -1 \end{aligned}$$

Solving this system of equations yields $C_1 = 2a/3 - 1$ and $C_2 = a$. Therefore,

$$\begin{aligned} y(t) &= \left(\frac{2a}{3} - 1\right) t e^{-2t/3} + a e^{-2t/3} \\ &= \left[\left(\frac{2a}{3} - 1\right) t + a\right] e^{-2t/3}. \end{aligned}$$

For $y(t)$ to always be positive, the quantity in square brackets must always be positive. More specifically, the coefficient of t must be positive. The value of a that separates positive and negative solutions is

$$\begin{aligned} \frac{2a}{3} - 1 &= 0 \\ a &= \frac{3}{2}. \end{aligned}$$