Problem 22

(a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}. \tag{i}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r, and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r} L[e^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \tag{ii}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t\exp(r_1 t)$ is also a solution of $L[y] = 0$.