Problem 27

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$xy'' - y' + 4x^3y = 0, \quad x > 0; \qquad y_1(x) = \sin x^2$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(x)$ is also a solution: $cy_1(x) = c\sin x^2$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of x.

$$y(x) = c(x)\sin x^2$$

Substitute this formula for $y(x)$ into the ODE.

$$x[c(x)\sin x^2]'' - [c(x)\sin x^2]' + 4x^3[c(x)\sin x^2] = 0$$

Evaluate the derivatives using the product rule.

$$x[c'(x)\sin x^2 + 2xc(x)\cos x^2]' - [c'(x)\sin x^2 + 2xc(x)\cos x^2] + 4x^3[c(x)\sin x^2] = 0$$

$$\begin{aligned} x[c''(x)\sin x^2 + 2xc'(x)\cos x^2 + 2c(x)\cos x^2 + 2xc'(x)\cos x^2 - 4x^2c(x)\sin x^2] \\ - [c'(x)\sin x^2 + 2xc(x)\cos x^2] + 4x^3[c(x)\sin x^2] = 0 \end{aligned}$$

$$\begin{aligned} xc''(x)\sin x^2 + 2x^2c'(x)\cos x^2 + \cancel{2xc(x)\cos x^2} + 2x^2c'(x)\cos x^2 - \cancel{4x^3c(x)\sin x^2} \\ - c'(x)\sin x^2 - \cancel{2xc(x)\cos x^2} + \cancel{4x^3c(x)\sin x^2} = 0 \end{aligned}$$

$$xc''(x)\sin x^2 + 4x^2c'(x)\cos x^2 - c'(x)\sin x^2 = 0$$

$$(x\sin x^2)c''(x) + (4x^2\cos x^2 - \sin x^2)c'(x) = 0$$

Solve for $c''(x)/c'(x)$.

$$(x\sin x^2)c''(x) = (-4x^2\cos x^2 + \sin x^2)c'(x)$$

$$\frac{c''(x)}{c'(x)} = -4x\frac{\cos x^2}{\sin x^2} + \frac{1}{x}$$

The left side can be written as $d/dx[\ln c'(x)]$ by the chain rule.

$$\frac{d}{dx}[\ln c'(x)] = -4x\frac{\cos x^2}{\sin x^2} + \frac{1}{x}$$

Integrate both sides with respect to x.

$$\begin{aligned} \ln c'(x) &= \int^x \left(-4s\frac{\cos s^2}{\sin s^2} + \frac{1}{s}\right) ds + C_1 \\ &= -\int^x 4s\frac{\cos s^2}{\sin s^2}\, ds + \int^x \frac{1}{s}\, ds + C_1 \end{aligned}$$

Make the substitution,

$$\begin{aligned} u &= \sin s^2 \\ du &= 2s\cos s^2\,ds, \end{aligned}$$

in the first integral and evaluate the second one.

$$\begin{aligned} \ln c'(x) &= -\int^{\sin x^2} 2\frac{du}{u} + \ln x + C_1 \\ &= -2\ln|u|\Big|^{\sin x^2} + \ln x + C_1 \\ &= -2\ln|\sin x^2| + \ln x + C_1 \\ &= \ln(\sin x^2)^{-2} + \ln x + C_1 \end{aligned}$$

Exponentiate both sides.

$$\begin{aligned} c'(x) &= e^{\ln(\sin x^2)^{-2} + \ln x + C_1} \\ &= e^{\ln(\sin x^2)^{-2}} e^{\ln x} e^{C_1} \\ &= (\sin x^2)^{-2} x e^{C_1} \\ &= \frac{x}{\sin^2 x^2} e^{C_1} \end{aligned}$$

Integrate both sides with respect to x once more.

$$c(x) = \int^x \frac{s}{\sin^2 s^2} e^{C_1}\,ds + C_2$$

Make the substitution,

$$\begin{aligned} v &= s^2 \\ dv &= 2s\,ds \quad \to \quad \frac{dv}{2} = s\,ds. \end{aligned}$$

As a result,

$$\begin{aligned} c(x) &= \int^{x^2} \frac{dv/2}{\sin^2 v} e^{C_1} + C_2 \\ &= \frac{e^{C_1}}{2} \int^{x^2} \csc^2 v\,dv + C_2 \\ &= -\frac{e^{C_1}}{2} \cot x^2 + C_2 \\ &= C_3 \cot x^2 + C_2, \end{aligned}$$

where a new constant C_3 was used for $-e^{C_1}/2$. Therefore, the general solution is

$$y(x) = C_3 \cos x^2 + C_2 \sin x^2;$$

the second solution is $\cos x^2$.