Problem 28

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(x)$ is also a solution: $cy_1(x) = ce^x$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of x.

$$y(x) = c(x)e^x$$

Substitute this formula for $y(x)$ into the ODE.

$$(x-1)[c(x)e^x]'' - x[c(x)e^x]' + c(x)e^x = 0$$

Evaluate the derivatives using the product rule.

$$(x-1)[c'(x)e^x + c(x)e^x]' - x[c'(x)e^x + c(x)e^x] + c(x)e^x = 0$$

$$(x-1)[c''(x)e^x + c'(x)e^x + c'(x)e^x + c(x)e^x] - x[c'(x)e^x + c(x)e^x] + c(x)e^x = 0$$

$$(x-1)e^x c''(x) + (x-2)e^x c'(x) = 0$$

Divide both sides by $(x-1)e^x$.

$$c''(x) + \frac{x-2}{x-1}c'(x) = 0$$

This is a linear first-order ODE for $c'(x)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{s-2}{s-1}\,ds\right) = e^{x-\ln(x-1)} = e^{x+\ln(x-1)^{-1}} = e^x e^{\ln(x-1)^{-1}} = \frac{e^x}{x-1}$$

Proceed with the multiplication.

$$\frac{e^x}{x-1}c''(x) + \frac{x-2}{(x-1)^2}e^x c'(x) = 0$$

The left side can be written as $d/dx[Ic'(x)]$ by the product rule.

$$\frac{d}{dx}\left[\frac{e^x}{x-1}c'(x)\right] = 0$$

Integrate both sides with respect to x.

$$\frac{e^x}{x-1}c'(x) = C_1$$

Solve for $c'(x)$.

$$c'(x) = C_1(x-1)e^{-x}$$

Integrate both sides with respect to x once more.

$$c(x) = -C_1 x e^{-x} + C_2$$

Therefore, using C_3 for $-C_1$, the general solution is

$$y(x) = C_3 x + C_2 e^x;$$

the second solution is $y_2(x) = x$.