Problem 29

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

$$x^2y'' - (x - 0.1875)y = 0, \quad x > 0; \qquad y_1(x) = x^{1/4}e^{2\sqrt{x}}$$

Solution

Because this ODE is homogeneous, any constant multiple of $y_1(x)$ is also a solution: $cy_1(x) = cx^{1/4}e^{2\sqrt{x}}$. According to the method of reduction of order, the general solution is obtained by allowing c to vary as a function of x.

$$y(x) = c(x)x^{1/4}e^{2\sqrt{x}}$$

Substitute this formula for $y(x)$ into the ODE.

$$x^2[c(x)x^{1/4}e^{2\sqrt{x}}]'' - (x - 0.1875)[c(x)x^{1/4}e^{2\sqrt{x}}] = 0$$

Evaluate the derivatives using the product rule.

$$x^2\left[c'(x)x^{1/4}e^{2\sqrt{x}} + \frac{c(x)e^{2\sqrt{x}}}{4x^{3/4}} + \frac{c(x)e^{2\sqrt{x}}}{x^{1/4}}\right]' - (x - 0.1875)c(x)x^{1/4}e^{2\sqrt{x}} = 0$$

$$x^2\left[c''(x)x^{1/4}e^{2\sqrt{x}} + \frac{c'(x)e^{2\sqrt{x}}}{4x^{3/4}} + \frac{c'(x)e^{2\sqrt{x}}}{x^{1/4}} + \frac{c'(x)e^{2\sqrt{x}}}{4x^{3/4}} - \frac{3c(x)e^{2\sqrt{x}}}{16x^{7/4}} + \cancel{\frac{c(x)e^{2\sqrt{x}}}{4x^{5/4}}}\right.$$
$$\left. + \frac{c'(x)e^{2\sqrt{x}}}{x^{1/4}} - \cancel{\frac{c(x)e^{2\sqrt{x}}}{4x^{5/4}}} + \frac{c(x)e^{2\sqrt{x}}}{x^{3/4}}\right] - (x - 0.1875)c(x)x^{1/4}e^{2\sqrt{x}} = 0$$

$$c''(x)x^{9/4}e^{2\sqrt{x}} + \frac{c'(x)x^{5/4}e^{2\sqrt{x}}}{2} + 2c'(x)x^{7/4}e^{2\sqrt{x}} - \cancel{\frac{3c(x)x^{1/4}e^{2\sqrt{x}}}{16}} + \cancel{c(x)x^{5/4}e^{2\sqrt{x}}}$$
$$- \cancel{c(x)x^{5/4}e^{2\sqrt{x}}} + \cancel{0.1875c(x)x^{1/4}e^{2\sqrt{x}}} = 0$$

$$c''(x)x^{9/4}e^{2\sqrt{x}} + \frac{c'(x)x^{5/4}e^{2\sqrt{x}}}{2} + 2c'(x)x^{7/4}e^{2\sqrt{x}} = 0$$

Divide both sides by $x^{9/4}e^{2\sqrt{x}}$.

$$c''(x) + \frac{c'(x)}{2x} + \frac{2c'(x)}{x^{1/2}} = 0$$

$$c''(x) + \left(\frac{1}{2x} + \frac{2}{x^{1/2}}\right)c'(x) = 0$$

Solve for $c''(x)/c'(x)$.

$$\frac{c''(x)}{c'(x)} = -\left(\frac{1}{2x} + \frac{2}{x^{1/2}}\right)$$

The left side can be written as $d/dx[\ln c'(x)]$ by the chain rule.

$$\frac{d}{dx}[\ln c'(x)] = -\left(\frac{1}{2x} + \frac{2}{x^{1/2}}\right)$$

Integrate both sides with respect to x.

$$\begin{aligned}\ln c'(x) &= \int^x -\left(\frac{1}{2r} + \frac{2}{r^{1/2}}\right) dr + C_1 \\ &= -\frac{1}{2}\int^x \frac{dr}{r} - 2\int^x r^{-1/2}\, dr + C_1 \\ &= -\frac{1}{2}\ln x - 4x^{1/2} + C_1 \\ &= \ln x^{-1/2} - 4x^{1/2} + C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}c'(x) &= e^{\ln x^{-1/2} - 4x^{1/2} + C_1} \\ &= e^{\ln x^{-1/2}} e^{-4x^{1/2}} e^{C_1} \\ &= x^{-1/2} e^{-4x^{1/2}} e^{C_1}\end{aligned}$$

Integrate both sides with respect to x once more.

$$c(x) = \int^x r^{-1/2} e^{-4r^{1/2}} e^{C_1}\, dr + C_2$$

Make the following substitution.

$$\begin{aligned}u &= r^{1/2} \\ du &= \frac{1}{2} r^{-1/2}\, dr \quad \rightarrow \quad 2\, du = r^{-1/2}\, dr\end{aligned}$$

As a result,

$$\begin{aligned}c(x) &= \int^{x^{1/2}} e^{-4u} e^{C_1} (2\, du) + C_2 \\ &= 2e^{C_1} \int^{x^{1/2}} e^{-4u}\, du + C_2 \\ &= -\frac{e^{C_1}}{2} e^{-4u}\Big|^{x^{1/2}} + C_2 \\ &= -\frac{e^{C_1}}{2} e^{-4x^{1/2}} + C_2 \\ &= C_3 e^{-4\sqrt{x}} + C_2,\end{aligned}$$

where a new constant C_3 was used for $-e^{C_1}/2$. The general solution is then

$$\begin{aligned}y(x) &= c(x) x^{1/4} e^{2\sqrt{x}} \\ &= C_3 x^{1/4} e^{-2\sqrt{x}} + C_2 x^{1/4} e^{2\sqrt{x}},\end{aligned}$$

which means the second solution is $y_2(x) = x^{1/4} e^{-2\sqrt{x}}$.