Problem 32

The method of Problem 20 can be extended to second order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use Abel's formula [Eq. (23) of Section 3.2] to determine y_2.

Solution

The Wronskian of y_1 and y_2 is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} = y_1 y_2'' - y_1'' y_2.$$

Because y_1 and y_2 are both solutions to the ODE, they satsify

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiply both sides of the first equation by $-y_2$ and both sides of the second equation by y_1.

$$-y_1'' y_2 - p(t)y_1' y_2 - q(t)y_1 y_2 = 0$$
$$y_1 y_2'' + p(t)y_1 y_2' + q(t)y_1 y_2 = 0$$

Add the respective sides of each equation.

$$y_1 y_2'' - y_1'' y_2 + p(t)y_1 y_2' - p(t)y_1' y_2 = 0$$

Factor $p(t)$.

$$y_1 y_2'' - y_1'' y_2 + p(t)(y_1 y_2' - y_1' y_2) = 0$$

This equation can be written in terms of the Wronskian as

$$W' + p(t)W = 0.$$

Solve it now.

$$W' = -p(t)W$$
$$\frac{W'}{W} = -p(t)$$
$$\frac{d}{dt}(\ln W) = -p(t)$$
$$\ln W = -\int^t p(s)\,ds$$
$$W(t) = \exp\left[-\int^t p(s)\,ds\right]$$

Now that the Wronskian is known, use its definition to obtain an ODE for y_2.

$$y_1 y_2' - y_1' y_2 = \exp\left[-\int^t p(s)\,ds\right]$$

Divide both sides by y_1.

$$y_2' - \frac{y_1'}{y_1}y_2 = \frac{1}{y_1}\exp\left[-\int^t p(s)\,ds\right]$$

Write the coefficient of y_2 as the derivative of a logarithm; this is possible by the chain rule.

$$y_2' - \frac{d}{dt}(\ln y_1)y_2 = \frac{1}{y_1}\exp\left[-\int^t p(s)\,ds\right]$$

This is a first-order linear inhomogeneous ODE for $y_2(t)$, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t -\frac{d}{dr}[\ln y_1(r)]\,dr\right) = e^{-\ln y_1(t)} = e^{\ln[y_1(t)]^{-1}} = \frac{1}{y_1(t)}$$

Proceed with the multiplication.

$$\frac{1}{y_1(t)}y_2' - \frac{1}{y_1(t)}\frac{d}{dt}(\ln y_1)y_2 = \frac{1}{[y_1(t)]^2}\exp\left[-\int^t p(s)\,ds\right]$$

The left side can be written as $d/dt(Iy_2)$ by the product rule.

$$\frac{d}{dt}\left[\frac{1}{y_1(t)}y_2\right] = \frac{1}{[y_1(t)]^2}\exp\left[-\int^t p(s)\,ds\right]$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\frac{1}{y_1(t)}y_2 = \int^t \frac{1}{[y_1(r)]^2}\exp\left[-\int^r p(s)\,ds\right]dr$$

Therefore, multiplying both sides by $y_1(t)$,

$$y_2(t) = y_1(t)\int^t \frac{1}{[y_1(r)]^2}\exp\left[-\int^r p(s)\,ds\right]dr.$$

Note that both of the lower limits of integration are arbitrary.