Problem 2

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + 2y' + 5y = 3\sin 2t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 5y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-2i)t}$ and $y_c = e^{(-1+2i)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{(-1-2i)t} + C_2 e^{(-1+2i)t} \\
&= C_1 e^{-t-2it} + C_2 e^{-t+2it} \\
&= C_1 e^{-t} e^{-2it} + C_2 e^{-t} e^{2it} \\
&= C_1 e^{-t}[\cos(-2t) + i\sin(-2t)] + C_2 e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1 e^{-t}[\cos(2t) - i\sin(2t)] + C_2 e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1 e^{-t}\cos 2t - iC_1 e^{-t}\sin 2t + C_2 e^{-t}\cos 2t + iC_2 e^{-t}\sin 2t \\
&= (C_1 + C_2)e^{-t}\cos 2t + (-iC_1 + iC_2)e^{-t}\sin 2t \\
&= C_3 e^{-t}\cos 2t + C_4 e^{-t}\sin 2t
\end{aligned}$$

The particular solution satisfies

$$y_p'' + 2y_p' + 5y_p = 3\sin 2t.$$

Since the inhomogeneous term is a sine function and both odd and even derivatives are present, we assume the solution is of the form $y_p(t) = A\cos 2t + B\sin 2t$. Substitute this into the ODE to determine A and B.

$$(A\cos 2t + B\sin 2t)'' + 2(A\cos 2t + B\sin 2t)' + 5(A\cos 2t + B\sin 2t) = 3\sin 2t$$

$$(-2A\sin 2t + 2B\cos 2t)' + 2(-2A\sin 2t + 2B\cos 2t) + 5(A\cos 2t + B\sin 2t) = 3\sin 2t$$
$$(-4A\cos 2t - 4B\sin 2t) + 2(-2A\sin 2t + 2B\cos 2t) + 5(A\cos 2t + B\sin 2t) = 3\sin 2t$$
$$-4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 5A\cos 2t + 5B\sin 2t = 3\sin 2t$$
$$(A + 4B)\cos 2t + (-4A + B)\sin 2t = 3\sin 2t$$

For this equation to be true, A and B must satisfy the following system of equations.

$$A + 4B = 0$$
$$-4A + B = 3$$

Solving it yields $A = -12/17$ and $B = 3/17$, which means

$$y_p(t) = -\frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t.$$

Therefore,

$$y(t) = C_3 e^{-t}\cos 2t + C_4 e^{-t}\sin 2t - \frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t.$$