Problem 4

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c' - 6y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + re^{rt} - 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + r - 6 = 0$$

$$(r + 3)(r - 2) = 0$$

$$r = \{-3, 2\}$$

Two solutions to equation (1) are then $y_c = e^{-3t}$ and $y_c = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-3t} + C_2e^{2t}$$

The particular solution satisfies

$$y_p'' + y_p' - 6y_p = 12e^{3t} + 12e^{-2t}.$$

Since the inhomogeneous term is a sum of exponential functions, we assume the solution is of the form $y_p(t) = Ae^{3t} + Be^{-2t}$. Substitute this into the ODE to determine A and B.

$$(Ae^{3t} + Be^{-2t})'' + (Ae^{3t} + Be^{-2t})' - 6(Ae^{3t} + Be^{-2t}) = 12e^{3t} + 12e^{-2t}$$

$$(3Ae^{3t} - 2Be^{-2t})' + (3Ae^{3t} - 2Be^{-2t}) - 6(Ae^{3t} + Be^{-2t}) = 12e^{3t} + 12e^{-2t}$$

$$(9Ae^{3t} + 4Be^{-2t}) + (3Ae^{3t} - 2Be^{-2t}) - 6(Ae^{3t} + Be^{-2t}) = 12e^{3t} + 12e^{-2t}$$

$$9Ae^{3t} + 4Be^{-2t} + 3Ae^{3t} - 2Be^{-2t} - 6Ae^{3t} - 6Be^{-2t} = 12e^{3t} + 12e^{-2t}$$

$$(6A)e^{3t} + (-4B)e^{-2t} = 12e^{3t} + 12e^{-2t}$$

For this equation to be true, A and B must satisfy the following system of equations.

$$6A = 12$$

$$-4B = 12$$

Solving it yields $A = 2$ and $B = -3$, which means

$$y_p(t) = 2e^{3t} - 3e^{-2t}.$$

Therefore,

$$y(t) = C_1e^{-3t} + C_2e^{2t} + 2e^{3t} - 3e^{-2t}.$$