Problem 5

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' - 2y' - 3y = -3te^{-t}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' - 3y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) - 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = \{-1, 3\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-t} + C_2 e^{3t}$$

The particular solution satisfies

$$y_p'' - 2y_p' - 3y_p = -3te^{-t}.$$

Since the inhomogeneous term is in terms of $y_c = e^{-t}$, we include an extra factor of t in the trial solution: $y_p(t) = t(A + Bt)e^{-t}$. Substitute this into the ODE to determine A and B.

$$[t(A+Bt)e^{-t}]'' - 2[t(A+Bt)e^{-t}]' - 3[t(A+Bt)e^{-t}] = -3te^{-t}$$

$$[(A+Bt)e^{-t} + Bte^{-t} - t(A+Bt)e^{-t}]' - 2[(A+Bt)e^{-t} + Bte^{-t} - t(A+Bt)e^{-t}] - 3[t(A+Bt)e^{-t}] = -3te^{-t}$$

$$\begin{aligned}&[Be^{-t} - (A+Bt)e^{-t} + Be^{-t} - Bte^{-t} - (A+Bt)e^{-t} - Bte^{-t} + t(A+Bt)e^{-t}] \\ &\qquad - 2[(A+Bt)e^{-t} + Bte^{-t} - t(A+Bt)e^{-t}] - 3[t(A+Bt)e^{-t}] = -3te^{-t}\end{aligned}$$

$$\begin{aligned}&Be^{-t} - Ae^{-t} - Bte^{-t} + Be^{-t} - Bte^{-t} - Ae^{-t} - Bte^{-t} - Bte^{-t} + Ate^{-t} + Bt^2e^{-t} \\ &\qquad - 2Ae^{-t} - 2Bte^{-t} - 2Bte^{-t} + 2Ate^{-t} + 2Bt^2e^{-t} - 3Ate^{-t} - 3Bt^2e^{-t} = -3te^{-t}\end{aligned}$$

$$2Be^{-t} - 8Bte^{-t} - 4Ae^{-t} = -3te^{-t}$$
$$(2B - 4A)e^{-t} + (-8B)te^{-t} = -3te^{-t}$$

For this equation to be true, A and B must satisfy the following system of equations.

$$\begin{aligned} 2B - 4A &= 0 \\ -8B &= -3 \end{aligned}$$

Solving it yields $A = 3/16$ and $B = 3/8$, which means

$$y_p(t) = t\left(\frac{3}{16} + \frac{3}{8}t\right)e^{-t}.$$

Therefore,

$$y(t) = C_1e^{-t} + C_2e^{3t} + t\left(\frac{3}{16} + \frac{3}{8}t\right)e^{-t}.$$