Problem 9

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$2y'' + 3y' + y = t^2 + 3\sin t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$2y_c'' + 3y_c' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^2e^{rt}) + 3(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$2r^2 + 3r + 1 = 0$$

$$(2r+1)(r+1) = 0$$

$$r = \left\{-1, -\frac{1}{2}\right\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{-t/2}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-t} + C_2e^{-t/2}$$

On the other hand, the particular solution satisfies

$$2y_p'' + 3y_p' + y_p = t^2 + 3\sin t.$$

The inhomogeneous term has two components, a monomial and a sine function. Since both odd and even derivatives are present, both sine and cosine need to be included in the trial solution. For the monomial, all powers of t leading up to and including t^2 must be included as well. The trial solution is thus $y_p(t) = A + Bt + Ct^2 + D\cos t + E\sin t$. Substitute this into the ODE to determine A, B, C, D, and E.

$$2(A+Bt+Ct^2+D\cos t+E\sin t)''+3(A+Bt+Ct^2+D\cos t+E\sin t)'+(A+Bt+Ct^2+D\cos t+E\sin t) = t^2+3\sin t$$

$$2(B+2Ct-D\sin t+E\cos t)'+3(B+2Ct-D\sin t+E\cos t)+(A+Bt+Ct^2+D\cos t+E\sin t) = t^2+3\sin t$$

$$2(2C-D\cos t-E\sin t)+3(B+2Ct-D\sin t+E\cos t)+(A+Bt+Ct^2+D\cos t+E\sin t) = t^2+3\sin t$$

$$4C-2D\cos t-2E\sin t+3B+6Ct-3D\sin t+3E\cos t+A+Bt+Ct^2+D\cos t+E\sin t = t^2+3\sin t$$

$(4C + 3B + A) + (6C + B)t + (C)t^2 + (-2D + 3E + D)\cos t + (-2E - 3D + E)\sin t = t^2 + 3\sin t$

For this equation to be true, A, B, C, D, and E must satisfy the following system of equations.

$$\begin{aligned} 4C + 3B + A &= 0 \\ 6C + B &= 0 \\ C &= 1 \\ -2D + 3E + D &= 0 \\ -2E - 3D + E &= 3 \end{aligned}$$

Solving it yields $A = 14$, $B = -6$, $C = 1$, $D = -9/10$, and $E = -3/10$, which means

$$y_p(t) = 14 - 6t + t^2 - \frac{9}{10}\cos t - \frac{3}{10}\sin t.$$

Therefore,

$$y(t) = C_1e^{-t} + C_2e^{-t/2} + 14 - 6t + t^2 - \frac{9}{10}\cos t - \frac{3}{10}\sin t.$$