Problem 10

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + y = 3\sin 2t + t\cos 2t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{-it} + C_2e^{it} \\
&= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\
&= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\
&= C_3\cos t + C_4\sin t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p = 3\sin 2t + t\cos 2t.$$

There are two terms on the right side. For the first one, since only even derivatives are present, we will include $A\sin 2t$ in the trial solution. For the second one, we will include $Bt\cos 2t + C\sin 2t$. The trial solution is thus $y_p(t) = A\sin 2t + Bt\cos 2t + C\sin 2t$. Substitute this into the ODE to determine A and B and C.

$$(A\sin 2t + Bt\cos 2t + C\sin 2t)'' + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(2A\cos 2t + B\cos 2t - 2Bt\sin 2t + 2C\cos 2t)' + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(-4A\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t - 4C\sin 2t) + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(-4A - 2B - 2B - 4C + A + C)\sin 2t + (-4B + B)t\cos 2t = 3\sin 2t + t\cos 2t$$

For this equation to be true, A and B and C must satisfy the following system of equations.

$$\begin{aligned} -4A - 2B - 2B - 4C + A + C &= 3 \\ -4B + B &= 1 \end{aligned}$$

Solving it yields $A + C = -5/9$ and $B = -1/3$, which means

$$\begin{aligned} y_p(t) &= A\sin 2t + Bt\cos 2t + C\sin 2t \\ &= (A + C)\sin 2t + Bt\cos 2t \\ &= -\frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t. \end{aligned}$$

Therefore,

$$y(t) = C_3\cos t + C_4\sin t - \frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t.$$