Problem 12

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$u'' + \omega_0^2 u = \cos \omega_0 t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + \omega_0^2 u_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + \omega_0^2 (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \omega_0^2 = 0$$

$$r = \{-i\omega_0, i\omega_0\}$$

Two solutions to equation (1) are then $u_c = e^{-i\omega_0 t}$ and $u_c = e^{i\omega_0 t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u_c(t) &= C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t} \\
&= C_1[\cos(-\omega_0 t) + i\sin(-\omega_0 t)] + C_2[\cos(\omega_0 t) + i\sin(\omega_0 t)] \\
&= C_1[\cos(\omega_0 t) - i\sin(\omega_0 t)] + C_2[\cos(\omega_0 t) + i\sin(\omega_0 t)] \\
&= C_1 \cos \omega_0 t - iC_1 \sin \omega_0 t + C_2 \cos \omega_0 t + iC_2 \sin \omega_0 t \\
&= (C_1 + C_2) \cos \omega_0 t + (-iC_1 + iC_2) \sin \omega_0 t \\
&= C_3 \cos \omega_0 t + C_4 \sin \omega_0 t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + \omega_0^2 u_p = \cos \omega_0 t.$$

Since only even derivatives are present, the trial solution would be $u_p(t) = A\cos \omega_0 t$; however, because $\cos \omega_0 t$ satisfies equation (1), an extra factor of t needs to be included. This introduces sine terms in the equation, though, so the trial solution will be $u_p(t) = t(A\cos \omega_0 t + B\sin \omega_0 t)$. Substitute this into the ODE to determine A and B.

$$[t(A\cos \omega_0 t + B\sin \omega_0 t)]'' + \omega_0^2[t(A\cos \omega_0 t + B\sin \omega_0 t)] = \cos \omega_0 t$$

$$[(A\cos \omega_0 t + B\sin \omega_0 t) + t(-A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t)]' + \omega_0^2[t(A\cos \omega_0 t + B\sin \omega_0 t)] = \cos \omega_0 t$$

$$[(-A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t) + (-A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t) + t(-A\omega_0^2 \cos\omega_0 t - B\omega_0^2 \sin\omega_0 t)] \\ + \omega_0^2[t(A\cos\omega_0 t + B\sin\omega_0 t)] = \cos\omega_0 t$$

$$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t - \cancel{A\omega_0^2 t\cos\omega_0 t} - \cancel{B\omega_0^2 t\sin\omega_0 t} + \cancel{A\omega_0^2 t\cos\omega_0 t} + \cancel{B\omega_0^2 t\sin\omega_0 t} = \cos\omega_0 t$$

For this equation to be true, A and B must satisfy the following system of equations.

$$-2A\omega_0 = 0$$
$$2B\omega_0 = 1$$

Solving it yields $A = 0$ and $B = 1/(2\omega_0)$.

$$u_p(t) = t\left(\frac{1}{2\omega_0}\sin\omega_0 t\right)$$

Therefore,

$$u(t) = C_3\cos\omega_0 t + C_4\sin\omega_0 t + t\left(\frac{1}{2\omega_0}\sin\omega_0 t\right).$$