Problem 14

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' - y' - 2y = \cosh 2t$$

Hint: $\cosh t = (e^t + e^{-t})/2$

Solution

Rewrite the right side in terms of exponential functions.

$$y'' - y' - 2y = \frac{e^{2t} + e^{-2t}}{2}$$

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - y_c' - 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = \{-1, 2\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-t} + C_2 e^{2t}$$

On the other hand, the particular solution satisfies

$$y_p'' - y_p' - 2y_p = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}.$$

There are two terms on the right side. For the first one, we would include Ae^{2t} in the trial solution, but e^{2t} satisfies equation (1). An extra factor of t is needed then, so Ate^{2t} will be used instead. For the second term, we will use Be^{-2t}. The trial solution is thus $y_p(t) = Ate^{2t} + Be^{-2t}$. Substitute this into the ODE to determine A and B.

$$(Ate^{2t} + Be^{-2t})'' - (Ate^{2t} + Be^{-2t})' - 2(Ate^{2t} + Be^{-2t}) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

$$(Ae^{2t} + 2Ate^{2t} - 2Be^{-2t})' - (Ae^{2t} + 2Ate^{2t} - 2Be^{-2t}) - 2(Ate^{2t} + Be^{-2t}) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

$$(2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} + 4Be^{-2t}) - (Ae^{2t} + 2Ate^{2t} - 2Be^{-2t}) - 2(Ate^{2t} + Be^{-2t}) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

$$(2A + 2A - A)e^{2t} + (4A - 2A - 2A)te^{2t} + (4B + 2B - 2B)e^{-2t} = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

For this equation to be true, A and B must satisfy the following system of equations.

$$\begin{aligned} 2A + 2A - A &= \frac{1}{2} \\ 4B + 2B - 2B &= \frac{1}{2} \end{aligned}$$

Solving it yields $A = 1/6$ and $B = 1/8$, which means

$$y_p(t) = \frac{1}{6}te^{2t} + \frac{1}{8}e^{-2t}.$$

Therefore,

$$y(t) = C_1e^{-t} + C_2e^{2t} + \frac{1}{6}te^{2t} + \frac{1}{8}e^{-2t}.$$