Problem 15

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' + y' - 2y = 2t, \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c' - 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = \{-2, 1\}$$

Two solutions to equation (1) are then $y_c = e^{-2t}$ and $y_c = e^t$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-2t} + C_2e^t$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p' - 2y_p = 2t.$$

The trial solution we will use is $y_p(t) = A + Bt$, a polynomial that ends at the highest power of t. Substitute this into the ODE to determine A and B.

$$(A+Bt)'' + (A+Bt)' - 2(A+Bt) = 2t$$

$$(B)' + (B) - 2(A+Bt) = 2t$$

$$(0) + (B) - 2(A+Bt) = 2t$$

$$(B - 2A) + (-2B)t = 2t$$

For this equation to be true, A and B must satisfy the following system of equations.

$$B - 2A = 0$$

$$-2B = 2$$

Solving it yields $A = -1/2$ and $B = -1$, which means

$$y_p(t) = -\frac{1}{2} - t.$$

The general solution is then

$$y(t) = C_1 e^{-2t} + C_2 e^t - \frac{1}{2} - t.$$

Take a derivative of it with respect to t.

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t - 1$$

Apply the initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 - \frac{1}{2} = 0$$
$$y'(0) = -2C_1 + C_2 - 1 = 1$$

Solving this system of equations yields $C_1 = -1/2$ and $C_2 = 1$. Therefore,

$$y(t) = -\frac{1}{2}e^{-2t} + e^t - \frac{1}{2} - t.$$