Problem 23

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4) \sin t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 5y_c' + 6y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 5(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 5r + 6 = 0$$

$$(r - 2)(r - 3) = 0$$

$$r = \{2, 3\}$$

Two solutions to equation (1) are then $y_c = e^{2t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{2t} + C_2 e^{3t}$$

On the other hand, the particular solution satisfies

$$y_p'' - 5y_p' + 6y_p = e^t \cos 2t + e^{2t}(3t + 4) \sin t.$$

There are two terms on the right side. For the first one, we will include $e^t(A \cos 2t + B \sin 2t)$ in the trial solution. For the second one, we will include $e^{2t}(C + Dt)(E \cos t + F \sin t)$ in the trial solution. The trial solution is thus

$$y_p(t) = e^t(A \cos 2t + B \sin 2t) + e^{2t}(C + Dt)(E \cos t + F \sin t).$$

Substitute this into the ODE to determine A, B, C, D, E, and F.

$$\begin{aligned}
&[e^t(A\cos 2t + B\sin 2t) + e^{2t}(C + Dt)(E\cos t + F\sin t)]'' \\
&\qquad - 5[e^t(A\cos 2t + B\sin 2t) + e^{2t}(C + Dt)(E\cos t + F\sin t)]' \\
&\qquad\qquad + 6[e^t(A\cos 2t + B\sin 2t) + e^{2t}(C + Dt)(E\cos t + F\sin t)] \\
&\qquad\qquad\qquad = e^t\cos 2t + e^{2t}(3t + 4)\sin t
\end{aligned}$$

Evaluate the derivatives and fully simplify the left side. Also, expand the right side.

$$\begin{aligned}
&(-2A - 6B)e^t\cos 2t + (CE - 2DE - CF - DF)e^{2t}\sin t \\
&\qquad + (DE - DF)te^{2t}\sin t + (-CE - DE - CF + 2DF)e^{2t}\cos t \\
&\qquad + (-DE - DF)te^{2t}\cos t + (6A - 2B)e^t\sin 2t = e^t\cos 2t + 3te^{2t}\sin t + 4e^{2t}\sin t
\end{aligned}$$

For this equation to be true, A, B, C, D, E, and F must satisfy the following system of equations.

$$\begin{aligned}
-2A - 6B &= 1 \\
CE - 2DE - CF - DF &= 4 \\
DE - DF &= 3 \\
-CE - DE - CF + 2DF &= 0 \\
-DE - DF &= 0 \\
6A - 2B &= 0
\end{aligned}$$

Solving it yields $A = -1/20$, $B = -3/20$, $CE = 1/2$, $DE = 3/2$, $CF = -5$, and $DF = -3/2$, which means

$$\begin{aligned}
y_p(t) &= e^t\left(-\frac{1}{20}\cos 2t - \frac{3}{20}\sin 2t\right) + e^{2t}(C + Dt)(E\cos t + F\sin t) \\
&= -\frac{1}{20}e^t(\cos 2t + 3\sin 2t) + e^{2t}(CE\cos t + CF\sin t) + te^{2t}(DE\cos t + DF\sin t) \\
&= -\frac{1}{20}e^t(\cos 2t + 3\sin 2t) + e^{2t}\left(\frac{1}{2}\cos t - 5\sin t\right) + te^{2t}\left(\frac{3}{2}\cos t - \frac{3}{2}\sin t\right) \\
&= -\frac{1}{20}e^t(\cos 2t + 3\sin 2t) + \frac{1}{2}e^{2t}(\cos t - 10\sin t) + \frac{3}{2}te^{2t}(\cos t - \sin t).
\end{aligned}$$

Therefore,

$$y(t) = C_1e^{2t} + C_2e^{3t} - \frac{1}{20}e^t(\cos 2t + 3\sin 2t) + \frac{1}{2}e^{2t}(\cos t - 10\sin t) + \frac{3}{2}te^{2t}(\cos t - \sin t).$$