Problem 24

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$r = \{-1 - i, -1 + i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-i)t}$ and $y_c = e^{(-1+i)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{(-1-i)t} + C_2 e^{(-1+i)t} \\
&= C_1 e^{-t-it} + C_2 e^{-t+it} \\
&= C_1 e^{-t} e^{-it} + C_2 e^{-t} e^{it} \\
&= C_1 e^{-t}[\cos(-t) + i\sin(-t)] + C_2 e^{-t}[\cos(t) + i\sin(t)] \\
&= C_1 e^{-t}[\cos(t) - i\sin(t)] + C_2 e^{-t}[\cos(t) + i\sin(t)] \\
&= C_1 e^{-t}\cos t - iC_1 e^{-t}\sin t + C_2 e^{-t}\cos t + iC_2 e^{-t}\sin t \\
&= (C_1 + C_2)e^{-t}\cos t + (-iC_1 + iC_2)e^{-t}\sin t \\
&= C_3 e^{-t}\cos t + C_4 e^{-t}\sin t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + 2y_p = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t.$$

There are three terms on the right side. For the first one, we will include Ae^{-t} in the trial solution. For the second one, we would include $e^{-t}(B\cos t + C\sin t)$, but $e^{-t}\cos t$ and $e^{-t}\sin t$ satisfy equation (1). An extra factor of t is needed as a result. For the third one, we would include $e^{-t}(D + Et + Ft^2)(G\cos t + H\sin t)$, but an extra factor of t is needed since $e^{-t}\cos t$ and $e^{-t}\sin t$ are solutions of equation (1). The trial solution is thus

$$y_p(t) = Ae^{-t} + te^{-t}(B\cos t + C\sin t) + te^{-t}(D + Et + Ft^2)(G\cos t + H\sin t).$$

Substitute this into the ODE to determine A, B, C, D, E, F, G, and H.

$$\begin{aligned}[Ae^{-t} + te^{-t}(B\cos t + C\sin t) + te^{-t}(D + Et + Ft^2)(G\cos t + H\sin t)]'' \\ + 2[Ae^{-t} + te^{-t}(B\cos t + C\sin t) + te^{-t}(D + Et + Ft^2)(G\cos t + H\sin t)]' \\ + 2[Ae^{-t} + te^{-t}(B\cos t + C\sin t) + te^{-t}(D + Et + Ft^2)(G\cos t + H\sin t)] \\ = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t\end{aligned}$$

Evaluate the derivatives and fully simplify the left side.

$$\begin{aligned}(A)e^{-t} + (2C + 2EG + 2DH)e^{-t}\cos t + (6FG + 4EH)te^{-t}\cos t \\ + (6FH)t^2e^{-t}\cos t + (-2B - 2DG + 2EH)e^{-t}\sin t + (-4EG + 6FH)te^{-t}\sin t \\ + (-6FG)t^2e^{-t}\sin t = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t\end{aligned}$$

For this equation to be true, A, B, C, D, E, F, G, and H must satisfy the following system of equations.

$$\begin{aligned}A &= 3 \\ 2C + 2EG + 2DH &= 2 \\ 6FG + 4EH &= 0 \\ 6FH &= 0 \\ -2B - 2DG + 2EH &= 0 \\ -4EG + 6FH &= 0 \\ -6FG &= 4\end{aligned}$$

Solving it yields $A = 3$, $B + DG = 1$, $C + DH = 1$, $EG = 0$, $FH = 0$, $EH = 1$, and $FG = -2/3$, which means

$$\begin{aligned}y_p(t) &= Ae^{-t} + te^{-t}(B\cos t + C\sin t) + te^{-t}(D + Et + Ft^2)(G\cos t + H\sin t) \\ &= Ae^{-t} + Bte^{-t}\cos t + Cte^{-t}\sin t + DGte^{-t}\cos t + DHte^{-t}\sin t \\ &\quad + EGt^2e^{-t}\cos t + EHt^2e^{-t}\sin t + FGt^3e^{-t}\cos t + FHt^3e^{-t}\sin t \\ &= Ae^{-t} + (B + DG)te^{-t}\cos t + (C + DH)te^{-t}\sin t + EGt^2e^{-t}\cos t \\ &\quad + EHt^2e^{-t}\sin t + FGt^3e^{-t}\cos t + FHt^3e^{-t}\sin t \\ &= 3e^{-t} + te^{-t}\cos t + te^{-t}\sin t + t^2e^{-t}\sin t - \frac{2}{3}t^3e^{-t}\cos t.\end{aligned}$$

Therefore,

$$y(t) = C_3e^{-t}\cos t + C_4e^{-t}\sin t + 3e^{-t} + te^{-t}\cos t + te^{-t}\sin t + t^2e^{-t}\sin t - \frac{2}{3}t^3e^{-t}\cos t.$$