Problem 25

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 4y_c' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = \{2\}$$

One solution to equation (1) is then $y_c = e^{2t}$. Use the method of reduction of order to determine the general solution: Plug in $y_c(t) = c(t)e^{2t}$ into equation (1).

$$[c(t)e^{2t}]'' - 4[c(t)e^{2t}]' + 4c(t)e^{2t} = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^{2t} + 2c(t)e^{2t}]' - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4c(t)e^{2t} = 0$$

$$[c''(t)e^{2t} + 2c'(t)e^{2t} + 2c'(t)e^{2t} + 4c(t)e^{2t}] - 4[c'(t)e^{2t} + 2c(t)e^{2t}] + 4c(t)e^{2t} = 0$$

$$c''(t)e^{2t} + \cancel{2c'(t)e^{2t}} + \cancel{2c'(t)e^{2t}} + \cancel{4c(t)e^{2t}} - \cancel{4c'(t)e^{2t}} - \cancel{8c(t)e^{2t}} + \cancel{4c(t)e^{2t}} = 0$$

$$c''(t)e^{2t} = 0$$

Divide both sides by e^{2t}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

As a result, the general solution to equation (1) is

$$y_c(t) = C_1 t e^{2t} + C_2 e^{2t}.$$

The particular solution satisfies

$$y_p'' - 4y_p' + 4y_p = 2t^2 + 4te^{2t} + t\sin 2t.$$

There are three terms on the right side. For the first one, we will include $A + Bt + Ct^2$ in the trial solution. For the second one, we would include $(D + Et)e^{2t}$, but both e^{2t} and te^{2t} satisfy equation (1); an extra factor of t^2 is needed to account for this. For the third term, we will include $(F + Gt)(H\cos 2t + I\sin 2t)$. The trial solution is thus

$$y_p(t) = A + Bt + Ct^2 + t^2(D + Et)e^{2t} + (F + Gt)(H\cos 2t + I\sin 2t).$$

Substitute this into the ODE to determine A, B, C, D, E, F, G, H, and I.

$$\begin{aligned}
&[A + Bt + Ct^2 + t^2(D + Et)e^{2t} + (F + Gt)(H\cos 2t + I\sin 2t)]'' \\
&\quad - 4[A + Bt + Ct^2 + t^2(D + Et)e^{2t} + (F + Gt)(H\cos 2t + I\sin 2t)]' \\
&\quad\quad + 4[A + Bt + Ct^2 + t^2(D + Et)e^{2t} + (F + Gt)(H\cos 2t + I\sin 2t)] \\
&\quad\quad\quad = 2t^2 + 4te^{2t} + t\sin 2t
\end{aligned}$$

Evaluate the derivatives and fully simplify the left side.

$$\begin{aligned}
&(4A - 4B + 2C) + (4B - 8C)t + (4C)t^2 + (2D)e^{2t} \\
&\quad + (6E)te^{2t} + (-4GH - 8FI + 4GI)\cos 2t + (-8GI)t\cos 2t \\
&\quad\quad + (8FH - 4GH - 4GI)\sin 2t + (8GH)t\sin 2t = 2t^2 + 4te^{2t} + t\sin 2t
\end{aligned}$$

For this equation to be true, A, B, C, D, E, F, G, H, and I must satisfy the following system of equations.

$$\begin{aligned}
4A - 4B + 2C &= 0 \\
4B - 8C &= 0 \\
4C &= 2 \\
2D &= 0 \\
6E &= 4 \\
-4GH - 8FI + 4GI &= 0 \\
-8GI &= 0 \\
8FH - 4GH - 4GI &= 0 \\
8GH &= 1
\end{aligned}$$

Solving it yields $A = 3/4$, $B = 1$, $C = 1/2$, $D = 0$, $E = 2/3$, $GH = 1/8$, $FI = -1/16$, $GI = 0$, and $FH = 1/16$, which means

$$\begin{aligned} y_p(t) &= \frac{3}{4} + t + \frac{1}{2}t^2 + t^2\left(\frac{2}{3}t\right)e^{2t} + (F + Gt)(H\cos 2t + I\sin 2t) \\ &= \frac{3}{4} + t + \frac{1}{2}t^2 + \frac{2}{3}t^3e^{2t} + FH\cos 2t + FI\sin 2t + GHt\cos 2t + GIt\sin 2t \\ &= \frac{3}{4} + t + \frac{1}{2}t^2 + \frac{2}{3}t^3e^{2t} + \frac{1}{16}\cos 2t - \frac{1}{16}\sin 2t + \frac{1}{8}t\cos 2t. \end{aligned}$$

Therefore,

$$y(t) = C_1te^{2t} + C_2e^{2t} + \frac{3}{4} + t + \frac{1}{2}t^2 + \frac{2}{3}t^3e^{2t} + \frac{1}{16}\cos 2t - \frac{1}{16}\sin 2t + \frac{1}{8}t\cos 2t.$$