Problem 28

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' + 5y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 2(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-2i)t}$ and $y_c = e^{(-1+2i)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{(-1-2i)t} + C_2e^{(-1+2i)t} \\
&= C_1e^{-t-2it} + C_2e^{-t+2it} \\
&= C_1e^{-t}e^{-2it} + C_2e^{-t}e^{2it} \\
&= C_1e^{-t}[\cos(-2t) + i\sin(-2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1e^{-t}[\cos(2t) - i\sin(2t)] + C_2e^{-t}[\cos(2t) + i\sin(2t)] \\
&= C_1e^{-t}\cos 2t - iC_1e^{-t}\sin 2t + C_2e^{-t}\cos 2t + iC_2e^{-t}\sin 2t \\
&= (C_1 + C_2)e^{-t}\cos 2t + (-iC_1 + iC_2)e^{-t}\sin 2t \\
&= C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + 5y_p = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$$

There are two terms on the right side. For the first one, we would include $(A+Bt)e^{-t}(C\cos 2t + D\sin 2t)$ in the trial solution, but since $e^{-t}\cos 2t$ and $e^{-t}\sin 2t$ satisfy equation (1), it's necessary to have an extra factor of t. For the second one, we will include $(E+Ft)e^{-2t}(G\cos t + H\sin t)$. The trial solution is thus

$$y_p(t) = t(A+Bt)e^{-t}(C\cos 2t + D\sin 2t) + (E+Ft)e^{-2t}(G\cos t + H\sin t).$$

Substitute this into the ODE to determine A, B, C, D, E, F, G, and H.

$$\begin{aligned} &[t(A+Bt)e^{-t}(C\cos 2t + D\sin 2t) + (E+Ft)e^{-2t}(G\cos t + H\sin t)]'' \\ &\quad + 2[t(A+Bt)e^{-t}(C\cos 2t + D\sin 2t) + (E+Ft)e^{-2t}(G\cos t + H\sin t)]' \\ &\quad\quad + 5[t(A+Bt)e^{-t}(C\cos 2t + D\sin 2t) + (E+Ft)e^{-2t}(G\cos t + H\sin t)] \\ &\quad\quad\quad = 3te^{-t}\cos 2t - 2te^{-2t}\cos t \end{aligned}$$

Evaluate the derivatives and fully simplify the left side.

$$\begin{aligned} &(4EG - 2FG - 2EH + 2FH)e^{-2t}\cos t + (4FG - 2FH)te^{-2t}\cos t + (2BC + 4AD)e^{-t}\cos 2t \\ &\quad + (8BD)te^{-t}\cos 2t + (2EG - 2FG + 4EH - 2FH)e^{-2t}\sin t + (2FG + 4FH)te^{-2t}\sin t \\ &\quad\quad + (-4AC + 2BD)e^{-t}\sin 2t + (-8BC)te^{-t}\sin 2t = 3te^{-t}\cos 2t - 2te^{-2t}\cos t \end{aligned}$$

For this equation to be true, A, B, C, D, E, F, G, and H must satisfy the following system of equations.

$$\begin{aligned} 4EG - 2FG - 2EH + 2FH &= 0 \\ 4FG - 2FH &= -2 \\ 2BC + 4AD &= 0 \\ 8BD &= 3 \\ 2EG - 2FG + 4EH - 2FH &= 0 \\ 2FG + 4FH &= 0 \\ -4AC + 2BD &= 0 \\ -8BC &= 0 \end{aligned}$$

Solving it yields $EG = -7/25$, $FG = -2/5$, $EH = 1/25$, $FH = 1/5$, $BC = 0$, $AD = 0$, $BD = 3/8$, and $AC = 3/16$, which means

$$\begin{aligned} y_p(t) &= t(A+Bt)e^{-t}(C\cos 2t + D\sin 2t) + (E+Ft)e^{-2t}(G\cos t + H\sin t) \\ &= ACte^{-t}\cos 2t + ADte^{-t}\sin 2t + BCt^2e^{-t}\cos 2t + BDt^2e^{-t}\sin 2t \\ &\quad + EGe^{-2t}\cos t + EHe^{-2t}\sin t + FGte^{-2t}\cos t + FHte^{-2t}\sin t \\ &= \frac{3}{16}te^{-t}\cos 2t + \frac{3}{8}t^2e^{-t}\sin 2t \\ &\quad - \frac{7}{25}e^{-2t}\cos t + \frac{1}{25}e^{-2t}\sin t - \frac{2}{5}te^{-2t}\cos t + \frac{1}{5}te^{-2t}\sin t. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t + \frac{3}{16}te^{-t}\cos 2t + \frac{3}{8}t^2e^{-t}\sin 2t \\ &\quad - \frac{7}{25}e^{-2t}\cos t + \frac{1}{25}e^{-2t}\sin t - \frac{2}{5}te^{-2t}\cos t + \frac{1}{5}te^{-2t}\sin t. \end{aligned}$$