Problem 30

Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t.$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + \lambda^2 y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + \lambda^2 (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \lambda^2 = 0$$

$$r = \{-i\lambda, i\lambda\}$$

Two solutions to equation (1) are then $y_c = e^{-i\lambda t}$ and $y_c = e^{i\lambda t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-i\lambda t} + C_2 e^{i\lambda t} \\
&= C_1[\cos(-\lambda t) + i\sin(-\lambda t)] + C_2[\cos(\lambda t) + i\sin(\lambda t)] \\
&= C_1[\cos(\lambda t) - i\sin(\lambda t)] + C_2[\cos(\lambda t) + i\sin(\lambda t)] \\
&= C_1 \cos\lambda t - iC_1 \sin\lambda t + C_2\cos\lambda t + iC_2 \sin\lambda t \\
&= (C_1 + C_2)\cos\lambda t + (-iC_1 + iC_2)\sin\lambda t \\
&= C_3 \cos\lambda t + C_4 \sin\lambda t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + \lambda^2 y_p = \sum_{m=1}^{N} a_m \sin m\pi t.$$

There are N terms on the right side. Since only even derivatives are present, we can include $A_1 \sin \pi t$ in the trial solution for the first term, $A_2 \sin 2\pi t$ for the second term, and so on. The trial solution is thus

$$y_p(t) = \sum_{m=1}^{N} A_m \sin m\pi t.$$

Substitute this into the ODE to determine A_1, A_2, $\ldots$, A_N.

$$\left(\sum_{m=1}^{N} A_m \sin m\pi t\right)'' + \lambda^2 \left(\sum_{m=1}^{N} A_m \sin m\pi t\right) = \sum_{m=1}^{N} a_m \sin m\pi t$$

The derivatives can be brought into the summand because the sum is only finite.

$$\sum_{m=1}^{N} (A_m \sin m\pi t)'' + \lambda^2 \sum_{m=1}^{N} A_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t$$

$$\sum_{m=1}^{N} (-m^2\pi^2 A_m \sin m\pi t) + \lambda^2 \sum_{m=1}^{N} A_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t$$

Now combine the two sums on the left side.

$$\sum_{m=1}^{N} (-m^2\pi^2 A_m \sin m\pi t) + \sum_{m=1}^{N} \lambda^2 A_m \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t$$

$$\sum_{m=1}^{N} (-m^2\pi^2 A_m \sin m\pi t + \lambda^2 A_m \sin m\pi t) = \sum_{m=1}^{N} a_m \sin m\pi t$$

$$\sum_{m=1}^{N} (-m^2\pi^2 A_m + \lambda^2 A_m) \sin m\pi t = \sum_{m=1}^{N} a_m \sin m\pi t$$

The coefficients must be equal.

$$-m^2\pi^2 A_m + \lambda^2 A_m = a_m$$

Solve for A_m.

$$A_m = \frac{a_m}{\lambda^2 - m^2\pi^2}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= \sum_{m=1}^{N} A_m \sin m\pi t \\ &= \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t. \end{aligned}$$

Therefore,

$$y(t) = C_3 \cos \lambda t + C_4 \sin \lambda t + \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t.$$