Problem 32

Follow the instructions in Problem 31 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \pi/2, \\ 0, & t > \pi/2, \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solution

Solve the ODE on each time interval separately.

$$y'' + 2y' + 5y = 1, \quad 0 \le t \le \pi/2 \qquad y'' + 2y' + 5y = 0, \quad t > \pi/2$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation. It is the same for both intervals.

$$y_c'' + 2y_c' + 5y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 5(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to equation (1) are then $y_c = e^{(-1-2i)t}$ and $y_c = e^{(-1+2i)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(t) &= C_1 e^{(-1-2i)t} + C_2 e^{(-1+2i)t} \\ &= C_1 e^{-t-2it} + C_2 e^{-t+2it} \\ &= C_1 e^{-t} e^{-2it} + C_2 e^{-t} e^{2it} \\ &= C_1 e^{-t}[\cos(-2t) + i\sin(-2t)] + C_2 e^{-t}[\cos(2t) + i\sin(2t)] \\ &= C_1 e^{-t}[\cos(2t) - i\sin(2t)] + C_2 e^{-t}[\cos(2t) + i\sin(2t)] \\ &= C_1 e^{-t}\cos 2t - iC_1 e^{-t}\sin 2t + C_2 e^{-t}\cos 2t + iC_2 e^{-t}\sin 2t \\ &= (C_1 + C_2)e^{-t}\cos 2t + (-iC_1 + iC_2)e^{-t}\sin 2t \end{aligned}$$

Using new constants for the terms in parentheses, the complementary solution is

$$y_c(t) = \begin{cases} C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t & \text{if } 0 \leq t \leq \pi/2 \\ C_5e^{-t}\cos 2t + C_6e^{-t}\sin 2t & \text{if } t > \pi/2 \end{cases}.$$

On the other hand, the particular solution satisfies

$$y_p'' + 2y_p' + 5y_p = 1, \quad 0 \leq t \leq \pi/2 \qquad y_p'' + 2y_p' + 5y_p = 0, \quad t > \pi/2.$$

Since the inhomogeneous term for the first ODE is a monomial, the trial solution is a polynomial of terms leading up to and including the highest order: $y_p(t) = A$. The same is true for the second ODE: $y_p(t) = B$. Substitute these into the ODEs to determine A and B.

$$(A)'' + 2(A)' + 5(A) = 1, \quad 0 \leq t \leq \pi/2 \qquad (B)'' + 2(B)' + 5(B) = 0, \quad t > \pi/2$$

Simplify the left sides.

$$5A = 1, \quad 0 \leq t \leq \pi/2 \qquad 5B = 0, \quad t > \pi$$

For these equations to be true, the following system of equations must be satisfied.

$$\begin{aligned} 5A &= 1 \\ 5B &= 0 \end{aligned}$$

Solving it yields $A = 1/5$ and $B = 0$, which means

$$y_p(t) = \begin{cases} \frac{1}{5} & \text{if } 0 \leq t \leq \pi/2 \\ 0 & \text{if } t > \pi/2 \end{cases}.$$

The general solution is then

$$y(t) = \begin{cases} C_3e^{-t}\cos 2t + C_4e^{-t}\sin 2t + \frac{1}{5} & \text{if } 0 \leq t \leq \pi/2 \\ C_5e^{-t}\cos 2t + C_6e^{-t}\sin 2t & \text{if } t > \pi/2 \end{cases}.$$

Differentiate it once with respect to t.

$$y'(t) = \begin{cases} -C_3e^{-t}\cos 2t - 2C_3e^{-t}\sin 2t - C_4e^{-t}\sin 2t + 2C_4e^{-t}\cos 2t & \text{if } 0 \leq t \leq \pi/2 \\ -C_5e^{-t}\cos 2t - 2C_5e^{-t}\sin 2t - C_6e^{-t}\sin 2t + 2C_6e^{-t}\cos 2t & \text{if } t > \pi/2 \end{cases}$$

The initial conditions are prescribed at $t = 0$, so they can be used to determine C_3 and C_4.

$$\begin{aligned} y(0) &= C_3 + \frac{1}{5} = 0 \\ y'(0) &= -C_3 + 2C_4 = 0 \end{aligned}$$

Solving this system of equations yields $C_3 = -1/5$ and $C_4 = -1/10$.

$$y(t) = \begin{cases} -\frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t + \frac{1}{5} & \text{if } 0 \leq t \leq \pi/2 \\ C_5e^{-t}\cos 2t + C_6e^{-t}\sin 2t & \text{if } t > \pi/2 \end{cases}.$$

Use the fact that y and y' are continuous at $t = \pi/2$ to determine C_5 and C_6.

$$\begin{cases} \lim_{t \to \frac{\pi}{2}^-} y(t) = \lim_{t \to \frac{\pi}{2}^+} y(t) \\ \lim_{t \to \frac{\pi}{2}^-} y'(t) = \lim_{t \to \frac{\pi}{2}^+} y'(t) \end{cases}$$

$$\begin{cases} -\frac{1}{5}e^{-\pi/2}\cos\pi - \frac{1}{10}e^{-\pi/2}\sin\pi + \frac{1}{5} = C_5 e^{-\pi/2}\cos\pi + C_6 e^{-\pi/2}\sin\pi \\ \frac{1}{5}e^{-\pi/2}\cos\pi + \frac{2}{5}e^{-\pi/2}\sin\pi + \frac{1}{10}e^{-\pi/2}\sin\pi - \frac{1}{5}e^{-\pi/2}\cos\pi = -C_5 e^{-\pi/2}\cos\pi - 2C_5 e^{-\pi/2}\sin\pi - C_6 e^{-\pi/2}\sin\pi + 2C_6 e^{-\pi/2}\cos\pi \end{cases}$$

$$\begin{cases} \frac{1}{5}e^{-\pi/2} + \frac{1}{5} = -C_5 e^{-\pi/2} \\ 0 = C_5 e^{-\pi/2} - 2C_6 e^{-\pi/2} \end{cases}$$

$$\begin{cases} C_5 = -\frac{1}{5} - \frac{1}{5}e^{\pi/2} \\ C_6 = -\frac{1}{10} - \frac{1}{10}e^{\pi/2} \end{cases}$$

Therefore,

$$y(t) = \begin{cases} -\frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t + \frac{1}{5} & \text{if } 0 \le t \le \pi/2 \\ \left(-\frac{1}{5} - \frac{1}{5}e^{\pi/2}\right) e^{-t}\cos 2t + \left(-\frac{1}{10} - \frac{1}{10}e^{\pi/2}\right) e^{-t}\sin 2t & \text{if } t > \pi/2 \end{cases}.$$