Problem 35

In this problem we indicate an alternative procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{i}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

TYPO: In part (b), "Eq (i)" should read "Eq. (i)" to be consistent.

Solution

Part (a)

In order for the operator to be factored as indicated, it's necessary that

$$\begin{aligned} D^2 + bD + c &= (D - r_1)(D - r_2) \\ &= D(D - r_2) - r_1(D - r_2) \\ &= D^2 - D(r_2) - r_1 D + r_1 r_2 \\ &= D^2 - r_2 D - r_1 D + r_1 r_2 \\ &= D^2 + (-r_1 - r_2)D + r_1 r_2 \end{aligned}$$

$b = -r_1 - r_2$ and $c = r_1 r_2$.

[7] R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201. Also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal 8* (1977), pp. 132–141, for a more general discussion of factoring operators.

Part (b)

Here we will solve

$$y'' + by' + cy = g(t)$$

by using the method of operator factorization.

$$(D^2 + bD + c)y = g(t)$$

$$(D - r_1)(D - r_2)y = g(t)$$

Let $u = (D - r_2)y$. Then this equation becomes

$$(D - r_1)u = g(t).$$

By factoring the operator as we have, the original second-order ODE has been reduced to a (decoupled) system of first-order ODEs.

$$(D - r_1)u = g(t) \quad \to \quad u' - r_1 u = g(t) \tag{1}$$

$$(D - r_2)y = u(t) \quad \to \quad y' - r_2 y = u(t) \tag{2}$$

Begin by solving the first equation with an integrating factor I_1.

$$I_1 = \exp\left[\int^t (-r_1)\, ds\right] = e^{-r_1 t}$$

Multiply both sides of equation (1) by I_1.

$$e^{-r_1 t}u' - r_1 e^{-r_1 t}u = g(t)e^{-r_1 t}$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^{-r_1 t}u) = g(t)e^{-r_1 t}$$

Integrate both sides with respect to t.

$$e^{-r_1 t}u = \int^t g(s)e^{-r_1 s}\, ds + C_1$$

Multiply both sides by $e^{r_1 t}$.

$$\begin{aligned} u(t) &= e^{r_1 t}\int^t g(s)e^{-r_1 s}\, ds + C_1 e^{r_1 t} \\ &= \int^t g(s)e^{r_1(t-s)}\, ds + C_1 e^{r_1 t} \end{aligned}$$

Plug this formula for $u(t)$ into equation (2).

$$y' - r_2 y = \int^t g(s)e^{r_1(t-s)}\, ds + C_1 e^{r_1 t}$$

Use another integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left[\int^t (-r_2)\, ds\right] = e^{-r_2 t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-r_2t}y' - r_2e^{-r_2t}y = e^{-r_2t}\int^t g(s)e^{r_1(t-s)}\,ds + C_1e^{-r_2t}e^{r_1t}$$

The left side can be written as $d/dt(I_2y)$ by the product rule.

$$\frac{d}{dt}(e^{-r_2t}y) = e^{-r_2t}\int^t g(s)e^{r_1(t-s)}\,ds + C_1e^{(r_1-r_2)t}$$

Integrate both sides with respect to t.

$$e^{-r_2t}y = \int^t e^{-r_2q}\int^q g(s)e^{r_1(q-s)}\,ds\,dq + \frac{C_1}{r_1-r_2}e^{(r_1-r_2)t} + C_2$$

Multiply both sides by e^{r_2t}.

$$y(t) = e^{r_2t}\int^t e^{-r_2q}\int^q g(s)e^{r_1(q-s)}\,ds\,dq + \frac{C_1}{r_1-r_2}e^{r_2t}e^{(r_1-r_2)t} + C_2e^{r_2t}$$

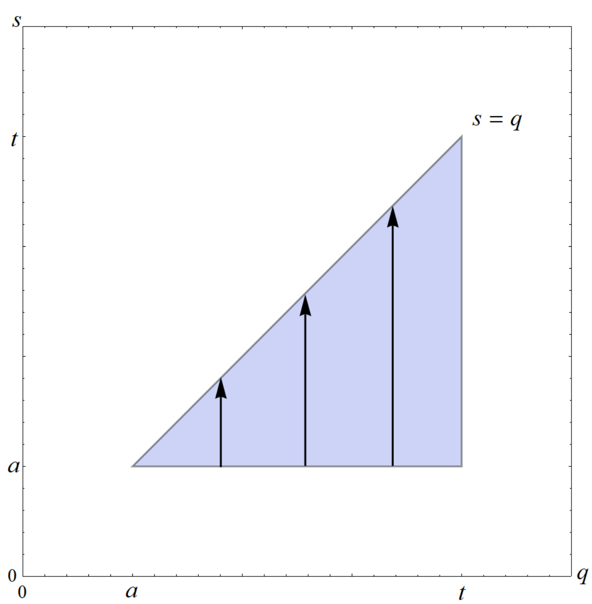
Use C_3 for $C_1/(r_1-r_2)$ and simplify.

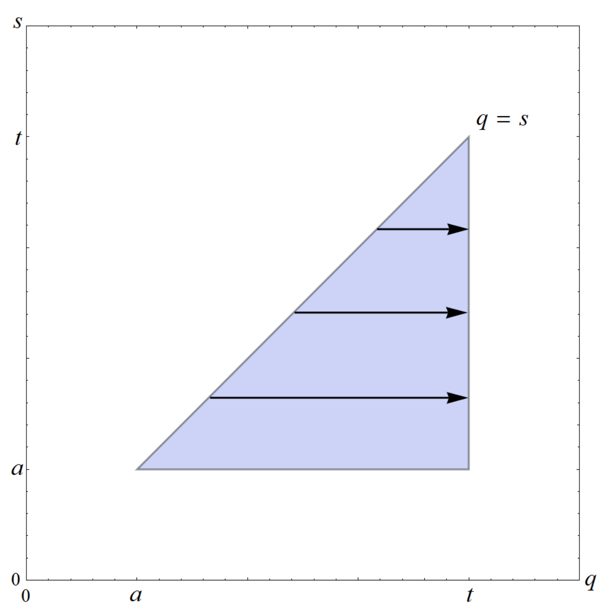
$$y(t) = \int^t\int^q g(s)e^{r_1(q-s)+r_2(t-q)}\,ds\,dq + C_3e^{r_1t} + C_2e^{r_2t}$$

Suppose that the initial conditions are prescribed at $t = a$: $y(a) = y_0$ and $y'(a) = v_0$. Then the lower limits of integration are both a.

$$y(t) = \int_a^t\int_a^q g(s)e^{r_1(q-s)+r_2(t-q)}\,ds\,dq + C_3e^{r_1t} + C_2e^{r_2t}$$

The domain of integration in the qs-plane is shown below on the left.





Integrate over this domain as shown on the right to switch the order of integration.

$$\begin{aligned}
y(t) &= \int_a^t \int_s^t g(s)e^{r_1(q-s)+r_2(t-q)}\,dq\,ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \int_a^t \left[\int_s^t g(s)e^{r_1q-r_1s+r_2t-r_2q}\,dq\right] ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \int_a^t g(s)e^{-r_1s+r_2t}\left[\int_s^t e^{(r_1-r_2)q}\,dq\right] ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \int_a^t g(s)e^{-r_1s+r_2t}\left[\frac{1}{r_1-r_2}e^{(r_1-r_2)q}\bigg|_s^t\right] ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \frac{1}{r_1-r_2}\int_a^t g(s)e^{-r_1s+r_2t}[e^{(r_1-r_2)t} - e^{(r_1-r_2)s}]\,ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \frac{1}{r_1-r_2}\int_a^t g(s)[e^{r_1t-\cancel{r_2t}-r_1s+\cancel{r_2t}} - e^{\cancel{r_1s}-r_2s-\cancel{r_1s}+r_2t}]\,ds + C_3e^{r_1t} + C_2e^{r_2t} \\
&= \frac{1}{r_1-r_2}\int_a^t g(s)[e^{r_1(t-s)} - e^{r_2(t-s)}]\,ds + C_3e^{r_1t} + C_2e^{r_2t}
\end{aligned}$$

Solving the system of equations,

$$\begin{aligned}
r_1 + r_2 &= -b \\
r_1r_2 &= c,
\end{aligned}$$

for r_1 and r_2 yields

$$r_1 = \frac{-b-\sqrt{b^2-4c}}{2} \quad \text{and} \quad r_2 = \frac{-b+\sqrt{b^2-4c}}{2}.$$

Therefore, changing back to the original variables, the general solution to Eq. (i) is

$$\boxed{\begin{aligned}
y(t) = \frac{1}{\sqrt{b^2-4c}}\int_a^t g(s)\left\{\exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right] - \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right]\right\} ds \\
+ C_3\exp\left(\frac{-b-\sqrt{b^2-4c}}{2}t\right) + C_2\exp\left(\frac{-b+\sqrt{b^2-4c}}{2}t\right),
\end{aligned}}$$

where, again, $t = a$ is when the initial conditions are given. Now we will check that this is in fact the solution. Differentiate it twice by using the Leibnitz rule, a more general version of the fundamental theorem of calculus.

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\,ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t}f(t,s)\,ds + \frac{dh}{dt}f[t,h(t)] - \frac{dg}{dt}f[t,g(t)]$$

The first derivative is

$$y'(t) = \frac{1}{\sqrt{b^2-4c}} \int_a^t g(s) \left\{ \frac{-b+\sqrt{b^2-4c}}{2} \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right] - \left(\frac{-b-\sqrt{b^2-4c}}{2}\right) \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right]\right\} ds$$
$$+ C_3 \left(\frac{-b-\sqrt{b^2-4c}}{2}\right) \exp\left(\frac{-b-\sqrt{b^2-4c}}{2}t\right) + C_2 \left(\frac{-b+\sqrt{b^2-4c}}{2}\right) \exp\left(\frac{-b+\sqrt{b^2-4c}}{2}t\right),$$

and the second derivative is

$$y''(t) = \frac{1}{\sqrt{b^2-4c}} \int_a^t g(s) \left\{ \left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right] - \left(\frac{-b-\sqrt{b^2-4c}}{2}\right)^2 \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right]\right\} ds$$
$$+ \frac{1}{\sqrt{b^2-4c}} g(t) \left[\frac{-b+\sqrt{b^2-4c}}{2} - \left(\frac{-b-\sqrt{b^2-4c}}{2}\right)\right]$$
$$+ C_3 \left(\frac{-b-\sqrt{b^2-4c}}{2}\right)^2 \exp\left(\frac{-b-\sqrt{b^2-4c}}{2}t\right) + C_2 \left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 \exp\left(\frac{-b+\sqrt{b^2-4c}}{2}t\right).$$

Consequently,

$$y'' + by' + cy = g(t) + \int_a^t \frac{g(s)}{\sqrt{b^2-4c}} \left\{ \left[\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b+\sqrt{b^2-4c}}{2}\right) + c\right] \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right]\right.$$
$$\left. + \left[-\left(\frac{-b-\sqrt{b^2-4c}}{2}\right)^2 - b\left(\frac{-b-\sqrt{b^2-4c}}{2}\right) - c\right] \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right]\right\} ds$$
$$+ C_3 \left[\left(\frac{-b-\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b-\sqrt{b^2-4c}}{2}\right) + c\right] \exp\left(\frac{-b-\sqrt{b^2-4c}}{2}t\right)$$
$$+ C_2 \left[\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b+\sqrt{b^2-4c}}{2}\right) + c\right] \exp\left(\frac{-b+\sqrt{b^2-4c}}{2}t\right)$$
$$= g(t).$$