Problem 37

In each of Problems 36 through 39, use the method of Problem 35 to solve the given differential equation.

$$2y'' + 3y' + y = t^2 + 3\sin t \qquad \text{(see Problem 9)}$$

Solution

Solve this ODE by the method of operator factorization.

$$\begin{aligned}
2y'' + 3y' + y &= t^2 + 3\sin t \\
2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y &= t^2 + 3\sin t \\
\left(2\frac{d^2}{dt^2} + 3\frac{d}{dt} + 1\right) y &= t^2 + 3\sin t \\
\left(2\frac{d}{dt} + 1\right)\left(\frac{d}{dt} + 1\right) y &= t^2 + 3\sin t
\end{aligned}$$

Let

$$u = \left(\frac{d}{dt} + 1\right) y.$$

Then the previous equation becomes

$$\left(2\frac{d}{dt} + 1\right) u = t^2 + 3\sin t.$$

As a result of factoring the operator, the original second-order ODE has reduced to a system of (decoupled) first-order ODEs.

$$\left(2\frac{d}{dt} + 1\right) u = t^2 + 3\sin t \qquad \rightarrow \qquad 2u' + u = t^2 + 3\sin t \tag{1}$$

$$\left(\frac{d}{dt} + 1\right) y = u(t) \qquad \rightarrow \qquad y' + y = u(t) \tag{2}$$

Begin by dividing both sides of equation (1) by 2.

$$u' + \frac{1}{2}u = \frac{1}{2}(t^2 + 3\sin t)$$

Use the integrating factor I_1 to solve it.

$$I_1 = \exp\left[\int^t \left(\frac{1}{2}\right) ds\right] = e^{t/2}$$

Multiply both sides of the previous equation by I_1.

$$e^{t/2}u' + \frac{1}{2}e^{t/2}u = \frac{e^{t/2}}{2}(t^2 + 3\sin t)$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^{t/2}u) = \frac{e^{t/2}}{2}(t^2 + 3\sin t)$$

Integrate both sides with respect to t, using integration by parts on the right side.

$$e^{t/2}u = e^{t/2}(8 - 4t + t^2) + \frac{3}{5}e^{t/2}(\sin t - 2\cos t) + C_1$$

Divide both sides by $e^{t/2}$.

$$u(t) = 8 - 4t + t^2 + \frac{3}{5}(\sin t - 2\cos t) + C_1 e^{-t/2}$$

Plug this result into equation (2).

$$y' + y = 8 - 4t + t^2 + \frac{3}{5}(\sin t - 2\cos t) + C_1 e^{-t/2}$$

Use another integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_2.

$$e^t y' + e^t y = 8e^t - 4te^t + t^2 e^t + \frac{3}{5}e^t(\sin t - 2\cos t) + C_1 e^{t/2}$$

The left side can be written as $d/dt(I_2 y)$ by the product rule.

$$\frac{d}{dt}(e^t y) = 8e^t - 4te^t + t^2 e^t + \frac{3}{5}e^t(\sin t - 2\cos t) + C_1 e^{t/2}$$

Integrate both sides with respect to t, using integration by parts on the right side.

$$e^t y = 14e^t - 6te^t + t^2 e^t - \frac{3}{10}e^t(3\cos t + \sin t) + 2C_1 e^{t/2} + C_2$$

Therefore, dividing both sides by e^t and using a new constant C_3 for $2C_1$,

$$y(t) = 14 - 6t + t^2 - \frac{3}{10}(3\cos t + \sin t) + C_3 e^{-t/2} + C_2 e^{-t}.$$