Problem 39

In each of Problems 36 through 39, use the method of Problem 35 to solve the given differential equation.

$$y'' + 2y' = 3 + 4\sin 2t \qquad \text{(see Problem 6)}$$

Solution

Solve this ODE by the method of operator factorization.

$$\begin{aligned} y'' + 2y' &= 3 + 4\sin 2t \\ \frac{d^2y}{dt^2} + 2\frac{dy}{dt} &= 3 + 4\sin 2t \\ \left(\frac{d^2}{dt^2} + 2\frac{d}{dt}\right) y &= 3 + 4\sin 2t \\ \frac{d}{dt}\left(\frac{d}{dt} + 2\right) y &= 3 + 4\sin 2t \end{aligned}$$

Let

$$u = \left(\frac{d}{dt} + 2\right) y.$$

Then the previous equation becomes

$$\frac{d}{dt} u = 3 + 4\sin 2t.$$

As a result of factoring the operator, the original second-order ODE has reduced to a system of (decoupled) first-order ODEs.

$$\frac{d}{dt} u = 3 + 4\sin 2t \qquad \rightarrow \qquad u' = 3 + 4\sin 2t \tag{1}$$

$$\left(\frac{d}{dt} + 2\right) y = u(t) \qquad \rightarrow \qquad y' + 2y = u(t) \tag{2}$$

Begin by integrating both sides of equation (1) with respect to t.

$$u(t) = 3t - 2\cos 2t + C_1$$

Plug this result into equation (2).

$$y' + 2y = 3t - 2\cos 2t + C_1$$

Use an integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left(\int^t 2\, ds\right) = e^{2t}$$

Multiply both sides of the previous equation by I_2.

$$e^{2t} y' + 2e^{2t} y = 3te^{2t} - 2e^{2t}\cos 2t + C_1 e^{2t}$$

The left side can be written as $d/dt(I_2 y)$ by the product rule.

$$\frac{d}{dt}(e^{2t}y) = 3te^{2t} - 2e^{2t}\cos 2t + C_1 e^{2t}$$

Integrate both sides with respect to t, using integration by parts on the right side.

$$e^{2t}y = -\frac{3}{4}e^{2t} + \frac{3}{2}te^{2t} - \frac{1}{2}e^{2t}(\cos 2t + \sin 2t) + \frac{C_1}{2}e^{2t} + C_2$$

Therefore, dividing both sides by e^{2t} and using C_3 for $C_1/2$,

$$y(t) = -\frac{3}{4} + \frac{3}{2}t - \frac{1}{2}(\cos 2t + \sin 2t) + C_3 + C_2 e^{-2t}.$$