Problem 1

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' - 2y' - 3y = 3e^{2t}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' - 3y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} - 2(re^{rt}) - 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = \{-1, 3\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-t} + C_2e^{3t}$$

The particular solution satisfies

$$y_p'' - 2y_p' - 3y_p = 3e^{2t}.$$

Since the inhomogeneous term is an exponential function, we assume the solution is of the form $y_p(t) = Ae^{2t}$. Substitute this into the ODE to determine A.

$$(Ae^{2t})'' - 2(Ae^{2t})' - 3(Ae^{2t}) = 3e^{2t}$$

$$4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$$

$$\cancel{4Ae^{2t}} - \cancel{4Ae^{2t}} - 3Ae^{2t} = 3e^{2t}$$

$$-3Ae^{2t} = 3e^{2t}$$

Dividing both sides by $-3e^{2t}$, we find that $A = -1$, which means

$$y_p(t) = -e^{2t}.$$

Therefore,

$$y(t) = C_1e^{-t} + C_2e^{3t} - e^{2t}.$$