# Problem 13

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + y' + 4y = 2\sinh t$$

*Hint:* $\sinh t = (e^t - e^{-t})/2$

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### Solution

Rewrite the right side in terms of exponential functions.

$$y'' + y' + 4y = e^t - e^{-t}$$

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + re^{rt} + 4(e^{rt}) = 0$$

Divide both sides by $e^{rt}$.

$$r^2 + r + 4 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2} = \frac{-1 \pm \sqrt{-15}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$$

$$r = \left\{-\frac{1}{2} - \frac{i\sqrt{15}}{2}, -\frac{1}{2} + \frac{i\sqrt{15}}{2}\right\}$$

Two solutions to equation (1) are then $y_c = e^{(-1/2 - i\sqrt{15}/2)t}$ and $y_c = e^{(-1/2 + i\sqrt{15}/2)t}$.

By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{(-1/2-i\sqrt{15}/2)t} + C_2e^{(-1/2+i\sqrt{15}/2)t} \\
&= C_1e^{-t/2-i\sqrt{15}t/2} + C_2e^{-t/2+i\sqrt{15}t/2} \\
&= C_1e^{-t/2}e^{-i\sqrt{15}t/2} + C_2e^{-t/2}e^{i\sqrt{15}t/2} \\
&= C_1e^{-t/2}\left[\cos\left(-\frac{\sqrt{15}}{2}t\right) + i\sin\left(-\frac{\sqrt{15}}{2}t\right)\right] + C_2e^{-t/2}\left[\cos\left(\frac{\sqrt{15}}{2}t\right) + i\sin\left(\frac{\sqrt{15}}{2}t\right)\right] \\
&= C_1e^{-t/2}\left[\cos\left(\frac{\sqrt{15}}{2}t\right) - i\sin\left(\frac{\sqrt{15}}{2}t\right)\right] + C_2e^{-t/2}\left[\cos\left(\frac{\sqrt{15}}{2}t\right) + i\sin\left(\frac{\sqrt{15}}{2}t\right)\right] \\
&= C_1e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) - iC_1e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + C_2e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + iC_2e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) \\
&= (C_1+C_2)e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + (-iC_1+iC_2)e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) \\
&= C_3e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + C_4e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p' + 4y_p = e^t - e^{-t}.$$

There are two terms on the right side. For the first one, we will include $Ae^t$ in the trial solution. For the second one, we will include $Be^{-t}$. The trial solution is thus $y_p(t) = Ae^t + Be^{-t}$. Substitute this into the ODE to determine $A$ and $B$.

$$(Ae^t + Be^{-t})'' + (Ae^t + Be^{-t})' + 4(Ae^t + Be^{-t}) = e^t - e^{-t}.$$

$$(Ae^t - Be^{-t})' + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) = e^t - e^{-t}.$$

$$(Ae^t + Be^{-t}) + (Ae^t - Be^{-t}) + 4(Ae^t + Be^{-t}) = e^t - e^{-t}.$$

$$(A + A + 4A)e^t + (B - B + 4B)e^{-t} = e^t - e^{-t}.$$

For this equation to be true, $A$ and $B$ must satisfy the following system of equations.

$$\begin{aligned}
A + A + 4A &= 1 \\
B - B + 4B &= -1
\end{aligned}$$

Solving it yields $A = 1/6$ and $B = -1/4$, which means

$$y_p(t) = \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

Therefore,

$$y(t) = C_3e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + C_4e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$