Problem 16

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' + 4y = t^2 + 3e^t, \qquad y(0) = 0, \quad y'(0) = 2$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4 = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to equation (1) are then $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1 e^{-2it} + C_2 e^{2it} \\
&= C_1[\cos(-2t) + i\sin(-2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1[\cos(2t) - i\sin(2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1\cos 2t - iC_1\sin 2t + C_2\cos 2t + iC_2\sin 2t \\
&= (C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t \\
&= C_3\cos 2t + C_4\sin 2t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + 4y_p = t^2 + 3e^t.$$

The right side has two terms. For the first one, we will include $A + Bt + Ct^2$, a polynomial that ends at the highest power of t, in the trial solution. For the second one, we will include De^t. The trial solution is thus $y_p(t) = A + Bt + Ct^2 + De^t$. Substitute this into the ODE to determine A, B, C, and D.

$$(A + Bt + Ct^2 + De^t)'' + 4(A + Bt + Ct^2 + De^t) = t^2 + 3e^t$$

$$(B + 2Ct + De^t)' + 4(A + Bt + Ct^2 + De^t) = t^2 + 3e^t$$

$$(2C + De^t) + 4(A + Bt + Ct^2 + De^t) = t^2 + 3e^t$$

$$(2C + 4A) + (4B)t + (4C)t^2 + (D + 4D)e^t = t^2 + 3e^t$$

For this equation to be true, A, B, C, and D must satisfy the following system of equations.

$$\begin{aligned} 2C + 4A &= 0 \\ 4B &= 0 \\ 4C &= 1 \\ D + 4D &= 3 \end{aligned}$$

Solving it yields $A = -1/8$, $B = 0$, $C = 1/4$, and $D = 3/5$, which means

$$y_p(t) = -\frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t.$$

The general solution is then

$$y(t) = C_3 \cos 2t + C_4 \sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t.$$

Take a derivative of it with respect to t.

$$y'(t) = -2C_3 \sin 2t + 2C_4 \cos 2t + \frac{1}{2}t + \frac{3}{5}e^t$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned} y(0) &= C_3 - \frac{1}{8} + \frac{3}{5} = 0 \\ y'(0) &= 2C_4 + \frac{3}{5} = 2 \end{aligned}$$

Solving this system of equations yields $C_3 = -19/40$ and $C_4 = 7/10$. Therefore,

$$y(t) = -\frac{19}{40}\cos 2t + \frac{7}{10}\sin 2t - \frac{1}{8} + \frac{1}{4}t^2 + \frac{3}{5}e^t.$$