Problem 17

In each of Problems 15 through 20, find the solution of the given initial value problem.

$$y'' - 2y' + y = te^t + 4, \qquad y(0) = 1, \quad y'(0) = 1$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$
$$r = \{1\}$$

One solution to equation (1) is then $y_c = e^t$. Use the method of reduction of order to determine the general solution: Plug in $y_c(t) = c(t)e^t$ into equation (1).

$$[c(t)e^t]'' - 2[c(t)e^t]' + c(t)e^t = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^t + c(t)e^t]' - 2[c'(t)e^t + c(t)e^t] + c(t)e^t = 0$$

$$[c''(t)e^t + c'(t)e^t + c'(t)e^t + c(t)e^t] - 2[c'(t)e^t + c(t)e^t] + c(t)e^t = 0$$

$$c''(t)e^t + \cancel{c'(t)e^t} + \cancel{c'(t)e^t} + \cancel{c(t)e^t} - \cancel{2c'(t)e^t} - \cancel{2c(t)e^t} + \cancel{c(t)e^t} = 0$$

$$c''(t)e^t = 0$$

Divide both sides by e^t.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

As a result, the general solution to equation (1) is

$$y_c(t) = C_1te^t + C_2e^t.$$

The particular solution satisfies

$$y_p'' - 2y_p' + y_p = te^t + 4.$$

There are two terms on the right side. For the first one, we would include $(A + Bt)e^t$ in the trial solution, but both e^t and te^t satisfy equation (1); an extra factor of t^2 is needed then to account for this. For the second one, we will include C. The trial solution is thus $y_p(t) = t^2(A + Bt)e^t + C$. Substitute this into the ODE to determine A and B and C.

$$[t^2(A + Bt)e^t + C]'' - 2[t^2(A + Bt)e^t + C]' + [t^2(A + Bt)e^t + C] = te^t + 4$$

$$[2t(A+Bt)e^t+Bt^2e^t+t^2(A+Bt)e^t]'-2[2t(A+Bt)e^t+Bt^2e^t+t^2(A+Bt)e^t]+[t^2(A+Bt)e^t+C] = te^t+4$$

$$\begin{aligned}[2(A + Bt)e^t &+ 2Bte^t + 2t(A + Bt)e^t + 2Bte^t + Bt^2e^t + 2t(A + Bt)e^t + Bt^2e^t + t^2(A + Bt)e^t] \\ &- 2[2t(A + Bt)e^t + Bt^2e^t + t^2(A + Bt)e^t] + [t^2(A + Bt)e^t + C] = te^t + 4\end{aligned}$$

$$\begin{aligned}2Ae^t &+ 4Bte^t + 2Ate^t + 2Bt^2e^t + 2Bte^t + Bt^2e^t + 2Ate^t + 2Bt^2e^t + Bt^2e^t + At^2e^t + Bt^3e^t \\ &- 4Ate^t - 4Bt^2e^t - 2Bt^2e^t - 2At^2e^t - 2Bt^3e^t + At^2e^t + Bt^3e^t + C = te^t + 4\end{aligned}$$

$$\begin{aligned}2Ae^t &+ 6Bte^t + \cancel{4Ate^t} + \cancel{6Bt^2e^t} + \cancel{At^2e^t} + Bt^3e^t \\ &- \cancel{4Ate^t} - \cancel{4Bt^2e^t} - \cancel{2Bt^2e^t} - \cancel{2At^2e^t} - 2Bt^3e^t + \cancel{At^2e^t} + Bt^3e^t + C = te^t + 4\end{aligned}$$

$$2Ae^t + 6Bte^t + C = te^t + 4$$

For this equation to be true, A and B and C must satisfy the following system of equations.

$$\begin{aligned}2A &= 0 \\ 6B &= 1 \\ C &= 4\end{aligned}$$

Solving it yields $A = 0$ and $B = 1/6$ and $C = 4$, which means

$$\begin{aligned}y_p(t) &= t^2\left(0 + \frac{1}{6}t\right)e^t + 4 \\ &= \frac{1}{6}t^3e^t + 4.\end{aligned}$$

The general solution is then

$$y(t) = C_1te^t + C_2e^t + \frac{1}{6}t^3e^t + 4.$$

Take a derivative of it with respect to t.

$$y'(t) = C_1e^t + C_1te^t + C_2e^t + \frac{1}{2}t^2e^t$$

Apply the initial conditions now to determine C_1 and C_2.

$$\begin{aligned}y(0) &= C_2 + 4 = 1 \\ y'(0) &= C_1 + C_2 = 1\end{aligned}$$

Solving this system of equations yields $C_1 = 4$ and $C_2 = -3$. Therefore,

$$y(t) = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4.$$