Problem 27

In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

$$y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 3y_c' + 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 3(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 3r + 2 = 0$$

$$(r + 2)(r + 1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to equation (1) are then $y_c = e^{-2t}$ and $y_c = e^{-t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-2t} + C_2 e^{-t}$$

On the other hand, the particular solution satisfies

$$y_p'' + 3y_p' + 2y_p = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t.$$

There are three terms on the right side. For the first one, we will include $e^t(A + Bt + Ct^2)(D\cos 2t + E\sin 2t)$ in the trial solution. For the second one, we will include $e^{-t}(F\cos t + G\sin t)$. For the third one, we will include He^t. The trial solution is thus

$$y_p(t) = e^t(A + Bt + Ct^2)(D\cos 2t + E\sin 2t) + e^{-t}(F\cos t + G\sin t) + He^t.$$

Substitute this into the ODE to determine A, B, C, D, E, F, G, and H.

$$\begin{aligned}[e^t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)+e^{-t}(F\cos t+G\sin t)+He^t]'' \\ +3[e^t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)+e^{-t}(F\cos t+G\sin t)+He^t]' \\ +2[e^t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)+e^{-t}(F\cos t+G\sin t)+He^t] \\ = e^t(t^2+1)\sin 2t+3e^{-t}\cos t+4e^t\end{aligned}$$

Evaluate the derivatives and fully simplify the left side. Also, expand the right side.

$$\begin{aligned}(-F+G)e^{-t}\cos t+(-F-G)e^{-t}\sin t+(6H)e^t+(2AD+5BD+2CD+10AE+4BE)e^t\cos 2t \\ +(2BD+10CD+10BE+8CE)te^t\cos 2t+(2CD+10CE)t^2e^t\cos 2t \\ +(-10AD-4BD+2AE+5BE+2CE)e^t\sin 2t \\ +(-10BD-8CD+2BE+10CE)te^t\sin 2t \\ +(-10CD+2CE)t^2e^t\sin 2t = t^2e^t\sin 2t+e^t\sin 2t+3e^{-t}\cos t+4e^t\end{aligned}$$

For this equation to be true, A, B, C, D, E, F, G, and H must satisfy the following system of equations.

$$\begin{aligned}-F+G&=3\\ -F-G&=0\\ 6H&=4\\ 2AD+5BD+2CD+10AE+4BE&=0\\ 2BD+10CD+10BE+8CE&=0\\ 2CD+10CE&=0\\ -10AD-4BD+2AE+5BE+2CE&=1\\ -10BD-8CD+2BE+10CE&=0\\ -10CD+2CE&=1\end{aligned}$$

Solving it yields $AD=-4105/35\,152$, $BD=73/676$, $CD=-5/52$, $AE=-1233/35\,152$, $BE=10/169$, $CE=1/52$, $F=-3/2$, $G=3/2$, and $H=2/3$, which means

$$\begin{aligned}y_p(t)&=e^t(A+Bt+Ct^2)(D\cos 2t+E\sin 2t)+e^{-t}(F\cos t+G\sin t)+He^t\\ &=ADe^t\cos 2t+AEe^t\sin 2t+BDte^t\cos 2t+BEte^t\sin 2t+CDt^2e^t\cos 2t\\ &\qquad +CEt^2e^t\sin 2t+Fe^{-t}\cos t+Ge^{-t}\sin t+He^t\\ &=-\frac{4105}{35\,152}e^t\cos 2t-\frac{1233}{35\,152}e^t\sin 2t+\frac{73}{676}te^t\cos 2t+\frac{10}{169}te^t\sin 2t\\ &\qquad -\frac{5}{52}t^2e^t\cos 2t+\frac{1}{52}t^2e^t\sin 2t-\frac{3}{2}e^{-t}\cos t+\frac{3}{2}e^{-t}\sin t+\frac{2}{3}e^t.\end{aligned}$$

Therefore,

$$\begin{aligned}y(t)=C_1e^{-2t}+C_2e^{-t}-\frac{4105}{35\,152}e^t\cos 2t-\frac{1233}{35\,152}e^t\sin 2t\\ +\frac{73}{676}te^t\cos 2t+\frac{10}{169}te^t\sin 2t-\frac{5}{52}t^2e^t\cos 2t+\frac{1}{52}t^2e^t\sin 2t\\ -\frac{3}{2}e^{-t}\cos t+\frac{3}{2}e^{-t}\sin t+\frac{2}{3}e^t.\end{aligned}$$