Problem 3

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' - y' - 2y = -2t + 4t^2$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - y_c' - 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - re^{rt} - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = \{-1, 2\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{-t} + C_2 e^{2t}$$

The particular solution satisfies

$$y_p'' - y_p' - 2y_p = -2t + 4t^2.$$

Since the inhomogeneous term is a polynomial, we assume the solution is of the form $y_p(t) = A + Bt + Ct^2$. Substitute this into the ODE to determine A.

$$(A + Bt + Ct^2)'' - (A + Bt + Ct^2)' - 2(A + Bt + Ct^2) = -2t + 4t^2$$

$$(B + 2Ct)' - (B + 2Ct) - 2(A + Bt + Ct^2) = -2t + 4t^2$$

$$(2C) - (B + 2Ct) - 2(A + Bt + Ct^2) = -2t + 4t^2$$

$$2C - B - 2Ct - 2A - 2Bt - 2Ct^2 = -2t + 4t^2$$

$$(-2A - B + 2C) + (-2B - 2C)t + (-2C)t^2 = -2t + 4t^2$$

For this equation to be true, A and B and C must satisfy the following system of equations.

$$\begin{aligned} -2A - B + 2C &= 0 \\ -2B - 2C &= -2 \\ -2C &= 4 \end{aligned}$$

Solving it yields $A = -7/2$ and $B = 3$ and $C = -2$, which means

$$y_p(t) = -\frac{7}{2} + 3t - 2t^2.$$

Therefore,

$$y(t) = C_1 e^{-t} + C_2 e^{2t} - \frac{7}{2} + 3t - 2t^2.$$