Problem 36

In each of Problems 36 through 39, use the method of Problem 35 to solve the given differential equation.

$$y'' - 3y' - 4y = 3e^{2t} \qquad \text{(see Example 1)}$$

Solution

Solve this ODE by the method of operator factorization.

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ \frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y &= 3e^{2t} \\ \left(\frac{d^2}{dt^2} - 3\frac{d}{dt} - 4\right) y &= 3e^{2t} \\ \left(\frac{d}{dt} - 4\right)\left(\frac{d}{dt} + 1\right) y &= 3e^{2t} \end{aligned}$$

Let

$$u = \left(\frac{d}{dt} + 1\right) y.$$

Then the previous equation becomes

$$\left(\frac{d}{dt} - 4\right) u = 3e^{2t}.$$

As a result of factoring the operator, the original second-order ODE has reduced to a system of (decoupled) first-order ODEs.

$$\left(\frac{d}{dt} - 4\right) u = 3e^{2t} \quad \rightarrow \quad u' - 4u = 3e^{2t} \tag{1}$$

$$\left(\frac{d}{dt} + 1\right) y = u(t) \quad \rightarrow \quad y' + y = u(t) \tag{2}$$

Begin by solving equation (1) with an integrating factor I_1.

$$I_1 = \exp\left[\int^t (-4)\, ds\right] = e^{-4t}$$

Multiply both sides of equation (1) by I_1.

$$e^{-4t}y' - 4e^{-4t}u = 3e^{-2t}$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^{-4t}u) = 3e^{-2t}$$

Integrate both sides with respect to t.

$$e^{-4t}u = -\frac{3}{2}e^{-2t} + C_1$$

Multiply both sides by e^{4t}.

$$u(t) = -\frac{3}{2}e^{2t} + C_1 e^{4t}$$

Plug this result into equation (2).

$$y' + y = -\frac{3}{2}e^{2t} + C_1 e^{4t}$$

Use another integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_2.

$$e^t y' + e^t y = -\frac{3}{2}e^{3t} + C_1 e^{5t}$$

The left side can be written as $d/dt(I_2 y)$ by the product rule.

$$\frac{d}{dt}(e^t y) = -\frac{3}{2}e^{3t} + C_1 e^{5t}$$

Integrate both sides with respect to t.

$$e^t y = -\frac{1}{2}e^{3t} + \frac{C_1}{5}e^{5t} + C_2$$

Therefore, dividing both sides by e^t and using a new constant C_3 for $C_1/5$,

$$y(t) = -\frac{1}{2}e^{2t} + C_3 e^{4t} + C_2 e^{-t}.$$