Problem 38

In each of Problems 36 through 39, use the method of Problem 35 to solve the given differential equation.

$$y'' + 2y' + y = 2e^{-t} \qquad \text{(see Problem 8)}$$

Solution

Solve this ODE by the method of operator factorization.

$$\begin{aligned} y'' + 2y' + y &= 2e^{-t} \\ \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y &= 2e^{-t} \\ \left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 1\right) y &= 2e^{-t} \\ \left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} + 1\right) y &= 2e^{-t} \end{aligned}$$

Let

$$u = \left(\frac{d}{dt} + 1\right) y.$$

Then the previous equation becomes

$$\left(\frac{d}{dt} + 1\right) u = 2e^{-t}.$$

As a result of factoring the operator, the original second-order ODE has reduced to a system of (decoupled) first-order ODEs.

$$\left(\frac{d}{dt} + 1\right) u = 2e^{-t} \qquad \rightarrow \qquad u' + u = 2e^{-t} \tag{1}$$

$$\left(\frac{d}{dt} + 1\right) y = u(t) \qquad \rightarrow \qquad y' + y = u(t) \tag{2}$$

Begin by using an integrating factor I_1 to solve equation (1).

$$I_1 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of equation (1) by I_1.

$$e^t u' + e^t u = 2$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^t u) = 2$$

Integrate both sides with respect to t.

$$e^t u = 2t + C_1$$

Divide both sides by e^t.

$$u(t) = 2te^{-t} + C_1 e^{-t}$$

Plug this result into equation (2).

$$y' + y = 2te^{-t} + C_1 e^{-t}$$

Use another integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_2.

$$e^t y' + e^t y = 2t + C_1$$

The left side can be written as $d/dt(I_2 y)$ by the product rule.

$$\frac{d}{dt}(e^t y) = 2t + C_1$$

Integrate both sides with respect to t.

$$e^t y = t^2 + C_1 t + C_2$$

Therefore, dividing both sides by e^t,

$$y(t) = t^2 e^{-t} + C_1 te^{-t} + C_2 e^{-t}.$$