Problem 7

In each of Problems 1 through 14, find the general solution of the given differential equation.

$$y'' + 9y = t^2 e^{3t} + 6$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 9y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 9 = 0$$

$$r = \{-3i, 3i\}$$

Two solutions to equation (1) are then $y_c = e^{-3it}$ and $y_c = e^{3it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(t) &= C_1 e^{-3it} + C_2 e^{3it} \\ &= C_1[\cos(-3t) + i\sin(-3t)] + C_2[\cos(3t) + i\sin(3t)] \\ &= C_1[\cos(3t) - i\sin(3t)] + C_2[\cos(3t) + i\sin(3t)] \\ &= C_1\cos 3t - iC_1\sin 3t + C_2\cos 3t + iC_2\sin 3t \\ &= (C_1 + C_2)\cos 3t + (-iC_1 + iC_2)\sin 3t \\ &= C_3\cos 3t + C_4\sin 3t \end{aligned}$$

The particular solution satisfies

$$y_p'' + 9y_p = t^2 e^{3t} + 6.$$

The inhomogeneous term has two components, a constant and a product of a polynomial and an exponential function. Use the trial solution $y_p(t) = A + (B + Ct + Dt^2)e^{3t}$. Substitute this into the ODE to determine A and B and C and D.

$$[A + (B + Ct + Dt^2)e^{3t}]'' + 9[A + (B + Ct + Dt^2)e^{3t}] = t^2 e^{3t} + 6$$

$$[(C + 2Dt)e^{3t} + 3(B + Ct + Dt^2)e^{3t}]' + 9[A + (B + Ct + Dt^2)e^{3t}] = t^2 e^{3t} + 6$$

$$[2De^{3t} + 3(C + 2Dt)e^{3t} + 3(C + 2Dt)e^{3t} + 9(B + Ct + Dt^2)e^{3t}] + 9[A + (B + Ct + Dt^2)e^{3t}] = t^2 e^{3t} + 6$$

$$2De^{3t} + 6(C + 2Dt)e^{3t} + 18(B + Ct + Dt^2)e^{3t} + 9A = t^2e^{3t} + 6$$
$$[(2D + 6C + 18B) + t(12D + 18C) + t^2(18D)]e^{3t} + 9A = t^2e^{3t} + 6$$

For this equation to be true, A and B and C and D must satisfy the following system of equations.

$$\begin{aligned} 2D + 6C + 18B &= 0 \\ 12D + 18C &= 0 \\ 18D &= 1 \\ 9A &= 6 \end{aligned}$$

Solving it yields $A = 2/3$ and $B = 1/162$ and $C = -1/27$ and $D = 1/18$, which means

$$\begin{aligned} y_p(t) &= \frac{2}{3} + \left(\frac{1}{162} - \frac{1}{27}t + \frac{1}{18}t^2\right)e^{3t} \\ &= \frac{2}{3} + \frac{1}{162}(1 - 6t + 9t^2)e^{3t} \\ &= \frac{2}{3} + \frac{1}{162}(3t - 1)^2e^{3t}. \end{aligned}$$

Therefore,

$$y(t) = C_3 \cos 3t + C_4 \sin 3t + \frac{2}{3} + \frac{1}{162}(3t - 1)^2e^{3t}.$$