Problem 4

In each of Problems 1 through 4, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

$$4y'' - 4y' + y = 16e^{t/2}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$4y_c'' - 4y_c' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2e^{rt}) - 4(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$4r^2 - 4r + 1 = 0$$

$$(2r-1)^2 = 0$$

$$r = \left\{\frac{1}{2}\right\}$$

One solution to equation (1) is $y_c = e^{t/2}$. Use the method of reduction of order to obtain the general solution: Plug $y_c(t) = c(t)e^{t/2}$ into equation (1) to obtain an ODE for $c(t)$.

$$4[c(t)e^{t/2}]'' - 4[c(t)e^{t/2}]' + [c(t)e^{t/2}] = 0$$

Evaluate the derivatives.

$$4\left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right]' - 4\left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right] + [c(t)e^{t/2}] = 0$$

$$4\left[c''(t)e^{t/2} + \frac{1}{2}c'(t)e^{t/2} + \frac{1}{2}c'(t)e^{t/2} + \frac{1}{4}c(t)e^{t/2}\right] - 4\left[c'(t)e^{t/2} + \frac{1}{2}c(t)e^{t/2}\right] + [c(t)e^{t/2}] = 0$$

$$4c''(t)e^{t/2} + \cancel{4c'(t)e^{t/2}} + \cancel{c(t)e^{t/2}} - \cancel{4c'(t)e^{t/2}} - \cancel{2c(t)e^{t/2}} + \cancel{c(t)e^{t/2}} = 0$$

$$4c''(t)e^{t/2} = 0$$

Divide both sides by $4e^{t/2}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

As a result,

$$\begin{aligned} y_c(t) &= c(t)e^{t/2} \\ &= (C_1 t + C_2)e^{t/2} \\ &= C_1 te^{t/2} + C_2 e^{t/2}. \end{aligned}$$

<u>Method of Variation of Parameters</u>

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)te^{t/2} + C_2(t)e^{t/2}$$

Substitute this into the original ODE to determine $C_1(t)$ and $C_2(t)$.

$$4[C_1(t)te^{t/2} + C_2(t)e^{t/2}]'' - 4[C_1(t)te^{t/2} + C_2(t)e^{t/2}]' + [C_1(t)te^{t/2} + C_2(t)e^{t/2}] = 16e^{t/2}$$

Evaluate the derivatives.

$$\begin{aligned} &[C_1'(t)te^{-t} + C_1(t)e^{-t} - C_1(t)te^{-t} + C_2'(t)e^{-t} - C_2(t)e^{-t}]' \\ &\quad + 2[C_1'(t)te^{-t} + C_1(t)e^{-t} - C_1(t)te^{-t} + C_2'(t)e^{-t} - C_2(t)e^{-t}] + [C_1(t)te^{-t} + C_2(t)e^{-t}] = 3e^{-t} \end{aligned}$$

$$\begin{aligned} &[C_1''(t)te^{-t} + C_1'(t)e^{-t} - C_1'(t)te^{-t} + C_1'(t)e^{-t} - C_1(t)e^{-t} - C_1'(t)te^{-t} - C_1(t)e^{-t} + C_1(t)te^{-t} \\ &\qquad + C_2''(t)e^{-t} - C_2'(t)e^{-t} - C_2'(t)e^{-t} + C_2(t)e^{-t}] \\ &\quad + 2[C_1'(t)te^{-t} + C_1(t)e^{-t} - C_1(t)te^{-t} + C_2'(t)e^{-t} - C_2(t)e^{-t}] + [C_1(t)te^{-t} + C_2(t)e^{-t}] = 3e^{-t} \end{aligned}$$

Simplify the left side.

$$\begin{aligned} &C_1''(t)te^{-t} + C_1'(t)e^{-t} - \cancel{C_1'(t)te^{-t}} + \cancel{C_1'(t)e^{-t}} - C_1(t)e^{-t} - C_1'(t)te^{-t} - C_1(t)e^{-t} + \cancel{C_1(t)te^{-t}} \\ &\qquad + C_2''(t)e^{-t} - \cancel{C_2'(t)e^{-t}} - \cancel{C_2'(t)e^{-t}} + C_2(t)e^{-t} \\ &\quad + \cancel{2C_1'(t)te^{-t}} + 2C_1(t)e^{-t} - \cancel{2C_1(t)te^{-t}} + \cancel{2C_2'(t)e^{-t}} - 2C_2(t)e^{-t} + \cancel{C_1(t)te^{-t}} + C_2(t)e^{-t} = 3e^{-t} \end{aligned}$$

$$C_1''(t)te^{-t} + C_1'(t)e^{-t} - C_1'(t)te^{-t} + C_2''(t)e^{-t} = 3e^{-t}$$

Multiply both sides by e^t.

$$C_1''(t)t + C_1'(t) - C_1'(t)t + C_2''(t) = 3$$

If we set

$$C_1''(t)t + C_1'(t) - C_1'(t)t = 0, \tag{2}$$

then the previous equation reduces to

$$C_2''(t) = 3 \tag{3}$$

The aim now is to solve this system of equations. Solve equation (2) first for $C_1(t)$. Divide both sides of it by t.

$$C_1''(t) + \left(\frac{1}{t} - 1\right)C_1'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left[\int^t \left(\frac{1}{s} - 1\right) ds\right] = e^{\ln t - t} = te^{-t}$$

Multiply both sides of the previous equation by I_1.

$$te^{-t}C_1''(t) + e^{-t}(1-t)C_1'(t) = 0$$

The left side can be written as $d/dt[I_1 C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[te^{-t}C_1'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$te^{-t}C_1'(t) = 0$$

Divide both sides by te^{-t}.

$$C_1'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = 0$$

Now solve equation (3) for $C_2(t)$. Integrate both sides of it with respect to t, setting the integration constant to zero.

$$C_2'(t) = 3t$$

Integrate both sides of it with respect to t once more, setting the integration constant to zero.

$$C_2(t) = \frac{3}{2}t^2$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_1(t)te^{-t} + C_2(t)e^{-t} \\ &= \frac{3}{2}t^2e^{-t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1te^{-t} + C_2e^{-t} + \frac{3}{2}t^2e^{-t}. \end{aligned}$$

Method of Undetermined Coefficients

The particular solution satisfies

$$y_p'' + 2y_p' + y_p = 3e^{-t}$$

Since the inhomogeneous term is an exponential function, we would use $y_p(t) = Ae^{-t}$ for the trial solution. However, since e^{-t} satisfies equation (1), an extra factor of t is needed. Actually, since te^{-t} also satisfies equation (1), one more factor of t is needed. Use $y_p(t) = At^2e^{-t}$ for the trial solution. Substitute this into the ODE to determine A.

$$(At^2e^{-t})'' + 2(At^2e^{-t})' + (At^2e^{-t}) = 3e^{-t}$$

Evaluate the derivatives.

$$(2Ate^{-t} - At^2e^{-t})' + 2(2Ate^{-t} - At^2e^{-t}) + (At^2e^{-t}) = 3e^{-t}$$

$$(2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}) + 2(2Ate^{-t} - At^2e^{-t}) + (At^2e^{-t}) = 3e^{-t}$$

$$2Ae^{-t} - \cancel{2Ate^{-t}} - \cancel{2Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} + \cancel{At^2e^{-t}} = 3e^{-t}$$

$$2Ae^{-t} = 3e^{-t}$$

For this equation to be true, A must satisfy $2A = 3$, which means $A = 3/2$. That means the particular solution is

$$y_p(t) = \frac{3}{2}t^2e^{-t}.$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1te^{-t} + C_2e^{-t} + \frac{3}{2}t^2e^{-t}. \end{aligned}$$