Problem 5

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$y'' + y = \tan t, \qquad 0 < t < \pi/2$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$
$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y_c(t) &= C_1e^{-it} + C_2e^{it} \\ &= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\ &= C_3\cos t + C_4\sin t \end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)\cos t + C_4(t)\sin t$$

It satisfies the following ODE.

$$y_p'' + y_p = \tan t$$

Substitute the previous formula in for $y_p(t)$.

$$[C_3(t)\cos t + C_4(t)\sin t]'' + [C_3(t)\cos t + C_4(t)\sin t] = \tan t$$

Evaluate the derivatives.

$$[C_3'(t)\cos t - C_3(t)\sin t + C_4'(t)\sin t + C_4(t)\cos t]' + [C_3(t)\cos t + C_4(t)\sin t] = \tan t$$

$$[C_3''(t)\cos t - C_3'(t)\sin t - C_3'(t)\sin t - \cancel{C_3(t)\cos t} + C_4''(t)\sin t + C_4'(t)\cos t + C_4'(t)\cos t - \cancel{C_4(t)\sin t}] + [\cancel{C_3(t)\cos t} + \cancel{C_4(t)\sin t}] = \tan t$$

$$C_3''(t)\cos t - 2C_3'(t)\sin t + C_4''(t)\sin t + 2C_4'(t)\cos t = \tan t$$

If we set

$$C_4''(t)\sin t + 2C_4'(t)\cos t = 0, \tag{2}$$

then the previous equation reduces to

$$C_3''(t)\cos t - 2C_3'(t)\sin t = \tan t. \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by dividing equation (2) by $\sin t$.

$$C_4''(t) + 2\frac{\cos t}{\sin t}C_4'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t 2\frac{\cos s}{\sin s}\,ds\right) = e^{2\ln\sin t} = e^{\ln\sin^2 t} = \sin^2 t$$

Multiply both sides of the previous equation by I_1.

$$(\sin^2 t)C_4''(t) + (2\sin t\cos t)C_4'(t) = 0$$

The left side can be written as $d/dt[I_1C_4'(t)]$ by the product rule.

$$\frac{d}{dt}[(\sin^2 t)C_4'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\sin^2 t)C_4'(t) = 0$$

Divide both sides by $\sin^2 t$.

$$C_4'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_4(t) = 0$$

Divide both sides of equation (3) by $\cos t$.

$$C_3''(t) - 2\frac{\sin t}{\cos t}C_3'(t) = \frac{\sin t}{\cos^2 t}$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t -2\frac{\sin s}{\cos s}\,ds\right) = e^{2\ln\cos t} = e^{\ln\cos^2 t} = \cos^2 t$$

Multiply both sides of the previous equation by I_2.

$$(\cos^2 t)C_3''(t) - (2\sin t\cos t)C_3'(t) = \sin t$$

The left side can be written as $d/dt[I_2C_3'(t)]$ by the product rule.

$$\frac{d}{dt}[(\cos^2 t)C_3'(t)] = \sin t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\cos^2 t)C_3'(t) = -\cos t$$

Divide both sides by $\cos^2 t$.

$$C_3'(t) = -\sec t$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_3(t) = -\ln|\sec t + \tan t|$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_3(t)\cos t + C_4(t)\sin t \\ &= -(\cos t)\ln|\sec t + \tan t|. \end{aligned}$$

Note that the absolute value sign can be dropped since $0 < t < \pi/2$. Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_3\cos t + C_4\sin t - (\cos t)\ln(\sec t + \tan t). \end{aligned}$$