Problem 7

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \qquad t > 0$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = \{-2\}$$

One solution to equation (1) is $y_c = e^{-2t}$. Use the method of reduction of order to obtain the general solution: Plug $y_c(t) = c(t)e^{-2t}$ into equation (1) to obtain an ODE for $c(t)$.

$$[c(t)e^{-2t}]'' + 4[c(t)e^{-2t}]' + 4[c(t)e^{-2t}] = 0$$

Evaluate the derivatives.

$$[c'(t)e^{-2t} - 2c(t)e^{-2t}]' + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$

$$[c''(t)e^{-2t} - 2c'(t)e^{-2t} - 2c'(t)e^{-2t} + 4c(t)e^{-2t}] + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$

$$c''(t)e^{-2t} - \cancel{2c'(t)e^{-2t}} - \cancel{2c'(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} + \cancel{4c'(t)e^{-2t}} - \cancel{8c(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} = 0$$

$$c''(t)e^{-2t} = 0$$

Multiply both sides by e^{2t}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1t + C_2$$

As a result,

$$\begin{aligned} y_c(t) &= c(t)e^{-2t} \\ &= (C_1 t + C_2)e^{-2t} \\ &= C_1 te^{-2t} + C_2 e^{-2t}. \end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)te^{-2t} + C_2(t)e^{-2t}$$

It satisfies the following ODE.

$$y_p'' + 4y_p' + 4y_p = t^{-2}e^{-2t}$$

Substitute the previous formula in for $y_p(t)$.

$$[C_1(t)te^{-2t} + C_2(t)e^{-2t}]'' + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}]' + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t}$$

Evaluate the derivatives.

$$\begin{aligned} &[C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}]' \\ &\quad + 4[C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}] \\ &\quad + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t} \end{aligned}$$

$$\begin{aligned} &[C_1''(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1'(t)te^{-2t} + C_1'(t)e^{-2t} - 2C_1(t)e^{-2t} - 2C_1'(t)te^{-2t} - 2C_1(t)e^{-2t} + 4C_1(t)te^{-2t} \\ &\quad + C_2''(t)e^{-2t} - 2C_2'(t)e^{-2t} - 2C_2'(t)e^{-2t} + 4C_2(t)e^{-2t}] \\ &\quad + 4[C_1'(t)te^{-2t} + C_1(t)e^{-2t} - 2C_1(t)te^{-2t} + C_2'(t)e^{-2t} - 2C_2(t)e^{-2t}] \\ &\quad + 4[C_1(t)te^{-2t} + C_2(t)e^{-2t}] = t^{-2}e^{-2t} \end{aligned}$$

Simplify the left side.

$$\begin{aligned} &C_1''(t)te^{-2t} + C_1'(t)e^{-2t} - \cancel{2C_1'(t)te^{-2t}} + C_1'(t)e^{-2t} - \cancel{2C_1(t)e^{-2t}} - \cancel{2C_1'(t)te^{-2t}} - \cancel{2C_1(t)e^{-2t}} + 4C_1(t)te^{-2t} \\ &\quad + C_2''(t)e^{-2t} - \cancel{2C_2'(t)e^{-2t}} - \cancel{2C_2'(t)e^{-2t}} + 4C_2(t)e^{-2t} \\ &\quad + \cancel{4C_1'(t)te^{-2t}} + \cancel{4C_1(t)e^{-2t}} - 8C_1(t)te^{-2t} + \cancel{4C_2'(t)e^{-2t}} - 8C_2(t)e^{-2t} \\ &\quad + 4C_1(t)te^{-2t} + 4C_2(t)e^{-2t} = t^{-2}e^{-2t} \end{aligned}$$

$$C_1''(t)te^{-2t} + 2C_1'(t)e^{-2t} + C_2''(t)e^{-2t} = t^{-2}e^{-2t}$$

Multiply both sides by e^{2t}.

$$C_1''(t)t + 2C_1'(t) + C_2''(t) = t^{-2}$$

If we set

$$C_1''(t)t + 2C_1'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$C_2''(t) = t^{-2} \tag{3}$$

The aim now is to solve this system of equations. Solve equation (2) first for $C_1(t)$. Divide both sides of it by t.

$$C_1''(t) + \frac{2}{t}C_1'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t \frac{2}{s}\,ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the previous equation by I_1.

$$t^2 C_1''(t) + 2tC_1'(t) = 0$$

The left side can be written as $d/dt[I_1 C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[t^2 C_1'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$t^2 C_1'(t) = 0$$

Divide both sides by t^2.

$$C_1'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = 0$$

Now solve equation (3) for $C_2(t)$. Integrate both sides of it with respect to t, setting the integration constant to zero.

$$C_2'(t) = -t^{-1}$$

Integrate both sides of it with respect to t once more, setting the integration constant to zero.

$$C_2(t) = -\ln t$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_1(t)te^{-2t} + C_2(t)e^{-2t} \\ &= -e^{-2t}\ln t. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 te^{-2t} + C_2 e^{-2t} - e^{-2t}\ln t. \end{aligned}$$