# Problem 10

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, $g$ is an arbitrary continuous function.

$$y'' - 2y' + y = e^t/(1+t^2)$$

---

### Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by $e^{rt}$.

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = \{1\}$$

One solution to equation (1) is $y_c = e^t$. Use the method of reduction of order to obtain the general solution: Plug $y_c(t) = c(t)e^t$ into equation (1) to obtain an ODE for $c(t)$.

$$[c(t)e^t]'' - 2[c(t)e^t]' + [c(t)e^t] = 0$$

Evaluate the derivatives.

$$[c'(t)e^t + c(t)e^t]' - 2[c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$[c''(t)e^t + c'(t)e^t + c'(t)e^t + c(t)e^t] - 2[c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$c''(t)e^t + \cancel{c'(t)e^t} + \cancel{c'(t)e^t} + \cancel{c(t)e^t} - \cancel{2c'(t)e^t} - \cancel{2c(t)e^t} + \cancel{c(t)e^t} = 0$$

$$c''(t)e^t = 0$$

Divide both sides by $e^t$.

$$c''(t) = 0$$

Integrate both sides with respect to $t$.

$$c'(t) = C_1$$

Integrate both sides with respect to $t$ once more.

$$c(t) = C_1 t + C_2$$

As a result,

$$\begin{aligned} y_c(t) &= c(t)e^t \\ &= (C_1 t + C_2)e^t \\ &= C_1 t e^t + C_2 e^t. \end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)te^t + C_2(t)e^t$$

It satisfies the following ODE.

$$y_p'' - 2y_p' + y_p = \frac{e^t}{1+t^2}$$

Substitute the previous formula in for $y_p(t)$.

$$[C_1(t)te^t + C_2(t)e^t]'' - 2[C_1(t)te^t + C_2(t)e^t]' + [C_1(t)te^t + C_2(t)e^t] = \frac{e^t}{1+t^2}$$

Evaluate the derivatives.

$$\begin{aligned} &[C_1'(t)te^t + C_1(t)e^t + C_1(t)te^t + C_2'(t)e^t + C_2(t)e^t]' \\ &\qquad - 2[C_1'(t)te^t + C_1(t)e^t + C_1(t)te^t + C_2'(t)e^t + C_2(t)e^t] \\ &\qquad\qquad + [C_1(t)te^t + C_2(t)e^t] = \frac{e^t}{1+t^2} \end{aligned}$$

$$\begin{aligned} &[C_1''(t)te^t + C_1'(t)e^t + C_1'(t)te^t + C_1'(t)e^t + C_1(t)e^t + C_1'(t)te^t + C_1(t)e^t + C_1(t)te^t + C_2''(t)e^t + C_2'(t)e^t + C_2'(t)e^t + C_2(t)e^t] \\ &\qquad - 2[C_1'(t)te^t + C_1(t)e^t + C_1(t)te^t + C_2'(t)e^t + C_2(t)e^t] \\ &\qquad\qquad + [C_1(t)te^t + C_2(t)e^t] = \frac{e^t}{1+t^2} \end{aligned}$$

Simplify the left side.

$$\begin{aligned} &C_1''(t)te^t + C_1'(t)e^t + \cancel{C_1'(t)te^t} + C_1'(t)e^t + \cancel{C_1(t)e^t} + \cancel{C_1'(t)te^t} + \cancel{C_1(t)e^t} + \cancel{C_1(t)te^t} + C_2''(t)e^t + C_2'(t)e^t + C_2'(t)e^t + C_2(t)e^t \\ &\qquad - \cancel{2C_1'(t)te^t} - \cancel{2C_1(t)e^t} - \cancel{2C_1(t)te^t} - 2C_2'(t)e^t - 2C_2(t)e^t \\ &\qquad\qquad + \cancel{C_1(t)te^t} + C_2(t)e^t = \frac{e^t}{1+t^2} \end{aligned}$$

$$C_1''(t)te^t + 2C_1'(t)e^t + C_2''(t)e^t = \frac{e^t}{1+t^2}$$

Divide both sides by $e^t$.

$$C_1''(t)t + 2C_1'(t) + C_2''(t) = \frac{1}{1+t^2}$$

If we set

$$C_2''(t) = 0, \tag{2}$$

then the previous equation reduces to

$$C_1''(t)t + 2C_1'(t) = \frac{1}{1+t^2} \tag{3}$$

The aim now is to solve this system of equations. Solve equation (2) first for $C_2(t)$. Integrate both sides of it with respect to $t$, setting the integration constant to zero.

$$C_2'(t) = 0$$

Integrate both sides of it with respect to $t$ once more, setting the integration constant to zero.

$$C_2(t) = 0$$

Divide both sides of equation (3) by $t$.

$$C_1''(t) + \frac{2}{t}C_1'(t) = \frac{1}{t(1+t^2)}$$

Use an integrating factor $I_1$ to solve it.

$$I_1 = \exp\left(\int^t \frac{2}{s}\,ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the previous equation by $I_1$.

$$t^2C_1''(t) + 2tC_1'(t) = \frac{t}{1+t^2}$$

The left side can be written as $d/dt[I_1C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[t^2C_1'(t)] = \frac{t}{1+t^2}$$

Integrate both sides with respect to $t$, setting the integration constant to zero.

$$t^2C_1'(t) = \frac{1}{2}\ln(1+t^2)$$

Divide both sides by $t^2$.

$$C_1'(t) = \frac{1}{2t^2}\ln(1+t^2)$$

Integrate both sides with respect to $t$ once more, setting the integration constant to zero.

$$\begin{aligned}
C_1(t) &= \int^t \frac{1}{2s^2}\ln(1+s^2)\,ds \\
&= \frac{1}{2}\int^t \frac{d}{ds}\left(-\frac{1}{s}\right)\ln(1+s^2)\,ds \\
&= \frac{1}{2}\left[\left(-\frac{1}{s}\right)\ln(1+s^2)\Big|^t - \int^t \left(-\frac{1}{s}\right)\left(\frac{2s}{1+s^2}\right)ds\right] \\
&= \frac{1}{2}\left(-\frac{\ln(1+t^2)}{t} + 2\int^t \frac{ds}{1+s^2}\right) \\
&= -\frac{1}{2}\frac{\ln(1+t^2)}{t} + \tan^{-1} t \\
&= \tan^{-1} t - \frac{\ln\sqrt{1+t^2}}{t}
\end{aligned}$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_1(t)te^t + C_2(t)e^t \\ &= te^t \tan^{-1} t - e^t \ln \sqrt{1+t^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 te^t + C_2 e^t + te^t \tan^{-1} t - e^t \ln \sqrt{1+t^2}. \end{aligned}$$