Problem 11

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$y'' - 5y' + 6y = g(t)$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 5y_c' + 6y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 5(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 5r + 6 = 0$$

$$(r - 2)(r - 3) = 0$$

$$r = \{2, 3\}$$

Two solutions to equation (1) are then $y_c = e^{2t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{2t} + C_2 e^{3t}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{2t} + C_2(t)e^{3t}$$

It satisfies the following ODE.

$$y_p'' - 5y_p' + 6y_p = g(t)$$

Substitute the previous formula in for $y_p(t)$.

$$[C_1(t)e^{2t} + C_2(t)e^{3t}]'' - 5[C_1(t)e^{2t} + C_2(t)e^{3t}]' + 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = g(t)$$

Evaluate the derivatives.

$$[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}]' - 5[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}]$$
$$+ 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = g(t)$$

$$\begin{aligned}[C_1''(t)e^{2t} + 2C_1'(t)e^{2t} + 2C_1'(t)e^{2t} + 4C_1(t)e^{2t} + C_2''(t)e^{3t} + 3C_2'(t)e^{3t} + 3C_2'(t)e^{3t} + 9C_2(t)e^{3t}] \\ - 5[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}] \\ + 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = g(t)\end{aligned}$$

$$\begin{aligned}C_1''(t)e^{2t} + 2C_1'(t)e^{2t} + 2C_1'(t)e^{2t} + \cancel{4C_1(t)e^{2t}} + C_2''(t)e^{3t} + 3C_2'(t)e^{3t} + 3C_2'(t)e^{3t} + \cancel{9C_2(t)e^{3t}} \\ - 5C_1'(t)e^{2t} - \cancel{10C_1(t)e^{2t}} - 5C_2'(t)e^{3t} - \cancel{15C_2(t)e^{3t}} \\ + \cancel{6C_1(t)e^{2t}} + \cancel{6C_2(t)e^{3t}} = g(t)\end{aligned}$$

$$C_1''(t)e^{2t} - C_1'(t)e^{2t} + C_2''(t)e^{3t} + C_2'(t)e^{3t} = g(t)$$

If we set

$$C_2''(t)e^{3t} + C_2'(t)e^{3t} = 0, \tag{2}$$

then the previous equation reduces to

$$C_1''(t)e^{2t} - C_1'(t)e^{2t} = g(t). \tag{3}$$

The aim now is to solve this system of two equations for $C_1(t)$ and $C_2(t)$. Start by dividing equation (2) by e^{3t}.

$$C_2''(t) + C_2'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_1.

$$e^t C_2''(t) + e^t C_2'(t) = 0$$

The left side can be written as $d/dt[I_1 C_2'(t)]$ by the product rule.

$$\frac{d}{dt}[e^t C_2'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^t C_2'(t) = 0$$

Divide both sides by e^t.

$$C_2'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_2(t) = 0$$

Divide both sides of equation (3) by e^{2t}.

$$C_1''(t) - C_1'(t) = g(t)e^{-2t}$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t (-1)\, ds\right) = e^{-t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-t}C_1''(t) - e^{-t}C_1'(t) = g(t)e^{-3t}$$

The left side can be written as $d/dt[I_2C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[e^{-t}C_1'(t)] = g(t)e^{-3t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{-t}C_1'(t) = \int^t g(s)e^{-3s}\,ds$$

Multiply both sides by e^t.

$$C_1'(t) = e^t \int^t g(s)e^{-3s}\,ds$$

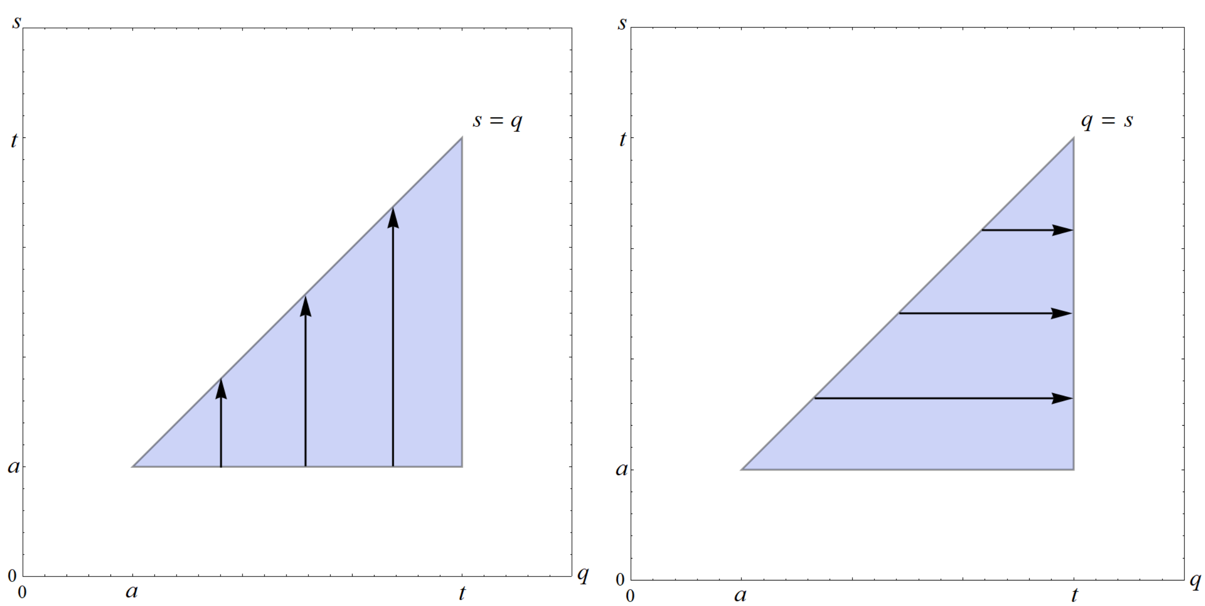
Integrate both sides with respect to t once more, setting the integration constant to zero.

$$\begin{aligned} C_1(t) &= \int^t e^q \int^q g(s)e^{-3s}\,ds\,dq \\ &= \int^t \int^q e^q g(s)e^{-3s}\,ds\,dq \end{aligned}$$

Suppose that the initial conditions are prescribed at $t = a$: $y(a) = y_0$ and $y'(a) = v_0$. Then the lower limits of integration are both a.

$$C_1(t) = \int_a^t \int_a^q e^q g(s)e^{-3s}\,ds\,dq$$

The domain of integration in the qs-plane is shown below on the left.



Integrate over this domain as shown on the right to switch the order of integration.

$$\begin{aligned} C_1(t) &= \int_a^t \int_s^t e^q g(s) e^{-3s}\, dq\, ds \\ &= \int_a^t g(s) e^{-3s} e^q \Big|_s^t \, ds \\ &= \int_a^t g(s) e^{-3s} (e^t - e^s)\, ds \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t) e^{2t} + C_2(t) e^{3t} \\ &= e^{2t} \int_a^t g(s) e^{-3s} (e^t - e^s)\, ds \\ &= \int_a^t g(s)(e^{3t-3s} - e^{2t-2s})\, ds \\ &= \int_a^t g(s)[e^{3(t-s)} - e^{2(t-s)}]\, ds. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 e^{2t} + C_2 e^{3t} + \int_a^t g(s)[e^{3(t-s)} - e^{2(t-s)}]\, ds, \end{aligned}$$

where, again, $t = a$ is when the initial conditions are given.