Problem 19

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$(1-x)y'' + xy' - y = g(x), \quad 0 < x < 1; \qquad y_1(x) = e^x, \quad y_2(x) = x$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$(1-x)y_1'' + xy_1' - y_1 \stackrel{?}{=} 0$$

$$(1-x)(e^x)'' + x(e^x)' - (e^x) \stackrel{?}{=} 0$$

$$(1-x)(e^x) + x(e^x) - (e^x) \stackrel{?}{=} 0$$

$$e^x - xe^x + xe^x - e^x \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$(1-x)y_2'' + xy_2' - y_2 \stackrel{?}{=} 0$$

$$(1-x)(x)'' + x(x)' - (x) \stackrel{?}{=} 0$$

$$x - x \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

$$y(x) = y_c(x) + y_p(x)$$

By the principle of superposition, $y_c(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

$$y_c(x) = C_1e^x + C_2x$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(x)$ to vary.

$$y_p(x) = C_1(x)e^x + C_2(x)x$$

It satisfies the following ODE.

$$(1-x)y_p'' + xy_p' - y_p = g(x)$$

Substitute the previous formula for $y_p(x)$.

$$(1-x)[C_1(x)e^x + C_2(x)x]'' + x[C_1(x)e^x + C_2(x)x]' - [C_1(x)e^x + C_2(x)x] = g(x)$$

Evaluate the derivatives.

$$(1-x)[C_1'(x)e^x + C_1(x)e^x + C_2'(x)x + C_2(x)]' + x[C_1'(x)e^x + C_1(x)e^x + C_2'(x)x + C_2(x)] \\ - [C_1(x)e^x + C_2(x)x] = g(x)$$

$$(1-x)[C_1''(x)e^x + C_1'(x)e^x + C_1'(x)e^x + C_1(x)e^x + C_2''(x)x + C_2'(x) + C_2'(x)] \\ + x[C_1'(x)e^x + C_1(x)e^x + C_2'(x)x + C_2(x)] - [C_1(x)e^x + C_2(x)x] = g(x)$$

$$e^xC_1''(x) - xe^xC_1''(x) - xe^xC_1'(x) + 2e^xC_1'(x) + xC_2''(x) - x^2C_2''(x) + x^2C_2'(x) - 2xC_2'(x) + 2C_2'(x) = g(x)$$

If we set

$$xC_2''(x) - x^2C_2''(x) + x^2C_2'(x) - 2xC_2'(x) + 2C_2'(x) = 0, \tag{1}$$

then the previous equation reduces to

$$e^xC_1''(x) - xe^xC_1''(x) - xe^xC_1'(x) + 2e^xC_1'(x) = g(x). \tag{2}$$

The aim now is to solve this system of equations for $C_1(x)$ and $C_2(x)$. Factor equation (1) and then divide both sides of it by $x - x^2$.

$$C_2''(x) + \frac{x^2 - 2x + 2}{x - x^2}C_2'(x) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^x \frac{s^2 - 2s + 2}{s - s^2}\,ds\right) = \exp\left[\int^x \left(\frac{2}{s} + \frac{s}{1-s}\right)ds\right] = e^{2\ln x - x - \ln(1-x)} = e^{\ln x^2}e^{-x}e^{\ln(1-x)^{-1}} = \frac{x^2}{1-x}e^{-x}$$

Multiply both sides of the previous equation by I_1.

$$\frac{x^2}{1-x}e^{-x}C_2''(x) + \frac{xe^{-x}(x^2 - 2x + 2)}{(1-x)^2}C_2'(x) = 0$$

The left side can be written as $d/dx[I_1C_2'(x)]$ by the product rule.

$$\frac{d}{dx}\left[\frac{x^2}{1-x}e^{-x}C_2'(x)\right] = 0$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$\frac{x^2}{1-x}e^{-x}C_2'(x) = 0$$

Divide both sides by $x^2(1-x)^{-1}e^{-x}$.

$$C_2'(x) = 0$$

Integrate both sides with respect to x once more, setting the integration constant to zero.

$$C_2(x) = 0$$

Factor the left side of equation (2)

$$e^x(1-x)C_1''(x) + e^x(2-x)C_1'(x) = g(x)$$

and then divide both sides by $e^x(1-x)$.

$$C_1''(x) + \frac{2-x}{1-x}C_1'(x) = \frac{g(x)}{e^x(1-x)}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^x \frac{2-s}{1-s}\,ds\right) = \exp\left[\int^x \left(\frac{2}{1-s} - \frac{s}{1-s}\right)ds\right] = e^{-2\ln(1-x)+x+\ln(1-x)} = e^{\ln(1-x)^{-1}}e^x = \frac{e^x}{1-x}$$

Multiply both sides of the previous equation by I_2.

$$\frac{e^x}{1-x}C_1''(x) + \frac{2-x}{(1-x)^2}e^x C_1'(x) = \frac{g(x)}{(1-x)^2}$$

The left side can be written as $d/dx[I_2C_1'(x)]$ by the product rule.

$$\frac{d}{dx}\left[\frac{e^x}{1-x}C_1'(x)\right] = \frac{g(x)}{(1-x)^2}$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$\frac{e^x}{1-x}C_1'(x) = \int^x \frac{g(s)}{(1-s)^2}\,ds$$

Divide both sides by $e^x(1-x)^{-1}$.

$$C_1'(x) = e^{-x}(1-x)\int^x \frac{g(s)}{(1-s)^2}\,ds$$

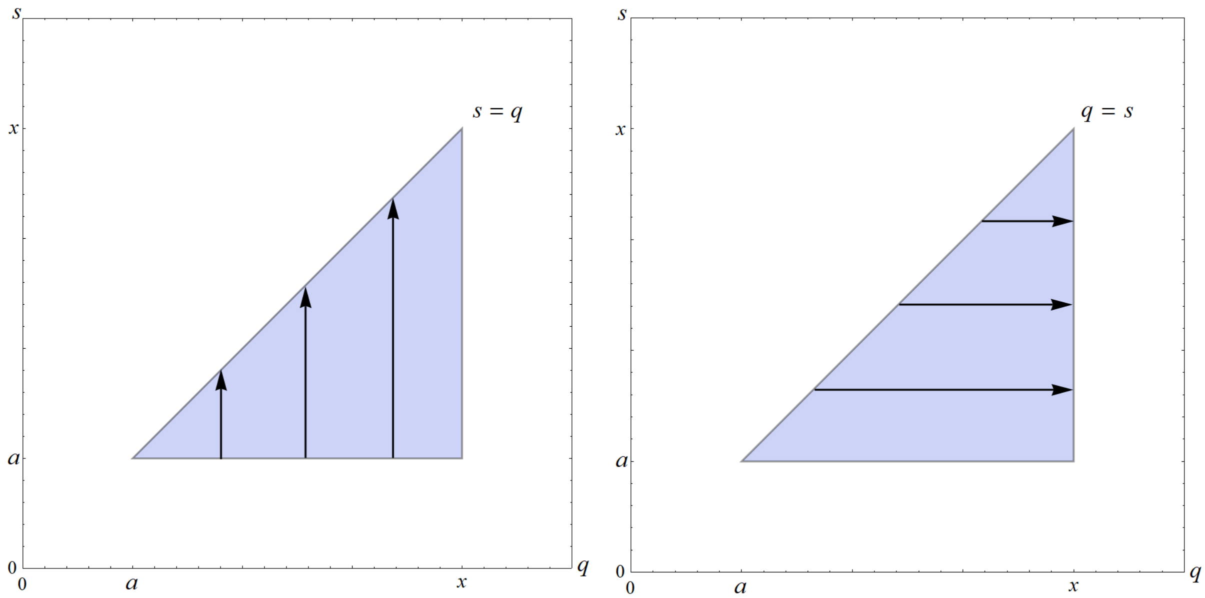
Integrate both sides with respect to x once more, setting the integration constant to zero.

$$\begin{aligned} C_1(t) &= \int^x e^{-q}(1-q)\int^q \frac{g(s)}{(1-s)^2}\,ds\,dq \\ &= \int^x \int^q e^{-q}(1-q)\frac{g(s)}{(1-s)^2}\,ds\,dq \end{aligned}$$

Suppose that the initial conditions are both given at $x = a$: $y(a) = y_0$ and $y'(a) = v_0$. Then the lower limits of integration are both a.

$$C_1(x) = \int_a^x \int_a^q e^{-q}(1-q)\frac{g(s)}{(1-s)^2}\,ds\,dq$$

The current mode of integration in the qs-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned} C_1(x) &= \int_a^x \int_s^x e^{-q}(1-q)\frac{g(s)}{(1-s)^2}\, dq\, ds \\ &= \int_a^x qe^{-q}\Big|_s^x \frac{g(s)}{(1-s)^2}\, ds \\ &= \int_a^x (xe^{-x} - se^{-s})\frac{g(s)}{(1-s)^2}\, ds \end{aligned}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(x)y_1(x) + C_2(x)y_2(x) \\ &= C_1(x)e^x + C_2(x)x \\ &= \left[\int_a^x (xe^{-x} - se^{-s})\frac{g(s)}{(1-s)^2}\, ds\right] e^x \\ &= \int_a^x (x - se^{x-s})\frac{g(s)}{(1-s)^2}\, ds. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(t) &= y_c(x) + y_p(x) \\ &= C_1 e^x + C_2 x + \int_a^x (x - se^{x-s})\frac{g(s)}{(1-s)^2}\, ds, \end{aligned}$$

where, again, $x = a$ is when the initial conditions are given.