Problem 21

Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{i}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \qquad u(t_0) = y_0, \qquad u'(t_0) = y_0', \tag{ii}$$

$$L[v] = g(t), \qquad v(t_0) = 0, \qquad v'(t_0) = 0, \tag{iii}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known.

Solution

Substitute $y(t) = u(t) + v(t)$ into Eq. (i).

$$[u(t) + v(t)]'' + p(t)[u(t) + v(t)]' + q(t)[u(t) + v(t)] = g(t)$$

Evaluate the derivatives.

$$[u''(t) + v''(t)] + p(t)[u'(t) + v'(t)] + q(t)[u(t) + v(t)] = g(t)$$

$$[u''(t) + p(t)u'(t) + q(t)u(t)] + [v''(t) + p(t)v'(t) + q(t)v(t)] = g(t)$$

If we set

$$u''(t) + p(t)u'(t) + q(t)u(t) = 0, \tag{1}$$

then the previous equation reduces to

$$v''(t) + p(t)v'(t) + q(t)v(t) = g(t). \tag{2}$$

Now use the substitution in the given initial conditions.

$$y(t_0) = y_0 \quad \rightarrow \quad u(t_0) + v(t_0) = y_0$$

$$y'(t_0) = y_0' \quad \rightarrow \quad u'(t_0) + v'(t_0) = y_0'$$

If we set $u(t_0) = y_0$ and $u'(t_0) = y_0'$, then we obtain the initial conditions satisfied by $v(t)$.

$$u(t_0) + v(t_0) = y_0 \quad \rightarrow \quad y_0 + v(t_0) = y_0 \quad \rightarrow \quad v(t_0) = 0$$

$$u'(t_0) + v'(t_0) = y_0' \quad \rightarrow \quad y_0' + v'(t_0) = y_0' \quad \rightarrow \quad v'(t_0) = 0.$$

Therefore, because the fully inhomogeneous initial value problem in Eq. (i) is linear, it can be split into two simpler initial value problems—one where the initial conditions are homogeneous and the second where the ODE is homogeneous. The general solution to Eq. (i) is the sum of the solutions to these simpler problems.