Problem 22

By choosing the lower limit of integration in Eq. (28) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)\, ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Thus Y can be identified with v in Problem 21.

Solution

Eq. (28) in the text can be written as follows.

$$\begin{aligned} Y(t) &= -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\, ds \qquad (28) \\ &= \int_{t_0}^{t} \frac{-y_1(t)y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds + \int_{t_0}^{t} \frac{y_1(s)y_2(t)g(s)}{W(y_1,y_2)(s)}\, ds \\ &= \int_{t_0}^{t} \left[\frac{-y_1(t)y_2(s)g(s)}{W(y_1,y_2)(s)} + \frac{y_1(s)y_2(t)g(s)}{W(y_1,y_2)(s)}\right] ds \\ &= \int_{t_0}^{t} \frac{y_1(s)y_2(t)g(s) - y_1(t)y_2(s)g(s)}{W(y_1,y_2)(s)}\, ds \\ &= \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(y_1,y_2)(s)} g(s)\, ds \end{aligned}$$

Note that the Wronskian of y_1 and y_2 is defined as

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Therefore,

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)\, ds.$$

Now we will solve the initial value problem,

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + p(t)y_c' + q(t)y_c = 0$$

Suppose that $y_1(t)$ and $y_2(t)$ satisfy this equation. Assuming that the Wronskian of y_1 and y_2 is not zero, the general solution is a linear combination of the two by the principle of superposition.

$$y_c(t) = C_1 y_1(t) + C_2 y_2(t)$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t) y_1(t) + C_2(t) y_2(t)$$

It satisfies the following ODE.

$$y_p'' + p(t) y_p' + q(t) y_p = g(t)$$

Plug in the previous formula for $y_p(t)$.

$$[C_1(t)y_1(t) + C_2(t)y_2(t)]'' + p(t)[C_1(t)y_1(t) + C_2(t)y_2(t)]' + q(t)[C_1(t)y_1(t) + C_2(t)y_2(t)] = g(t)$$

Evaluate the derivatives.

$$\begin{aligned}&[C_1'(t)y_1(t) + C_1(t)y_1'(t) + C_2'(t)y_2(t) + C_2(t)y_2'(t)]' \\ &\quad + p(t)[C_1'(t)y_1(t) + C_1(t)y_1'(t) + C_2'(t)y_2(t) + C_2(t)y_2'(t)] + q(t)[C_1(t)y_1(t) + C_2(t)y_2(t)] = g(t)\end{aligned}$$

$$\begin{aligned}&[C_1''(t)y_1(t) + C_1'(t)y_1'(t) + C_1'(t)y_1'(t) + C_1(t)y_1''(t) + C_2''(t)y_2(t) + C_2'(t)y_2'(t) + C_2'(t)y_2'(t) + C_2(t)y_2''(t)] \\ &\quad + p(t)[C_1'(t)y_1(t) + C_1(t)y_1'(t) + C_2'(t)y_2(t) + C_2(t)y_2'(t)] + q(t)[C_1(t)y_1(t) + C_2(t)y_2(t)] = g(t)\end{aligned}$$

$$\begin{aligned}&C_1''(t)y_1(t) + 2C_1'(t)y_1'(t) + C_2''(t)y_2(t) + 2C_2'(t)y_2'(t) + C_1(t)y_1''(t) + C_2(t)y_2''(t) \\ &\qquad + C_1'(t)p(t)y_1(t) + C_1(t)p(t)y_1'(t) + C_2'(t)p(t)y_2(t) + C_2(t)p(t)y_2'(t) \\ &\qquad\qquad + C_1(t)q(t)y_1(t) + C_2(t)q(t)y_2(t) = g(t)\end{aligned}$$

Since $y_1(t)$ and $y_2(t)$ satisfy the associated homogeneous equation, many terms on the left side disappear.

$$\begin{aligned}&C_1''(t)y_1(t) + 2C_1'(t)y_1'(t) + C_2''(t)y_2(t) + 2C_2'(t)y_2'(t) + C_1'(t)p(t)y_1(t) + C_2'(t)p(t)y_2(t) \\ &\qquad + C_1(t)[\underbrace{y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)}_{=0}] + C_2(t)[\underbrace{y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)}_{=0}] = g(t)\end{aligned}$$

$$C_1''(t)y_1(t) + 2C_1'(t)y_1'(t) + C_1'(t)p(t)y_1(t) + C_2''(t)y_2(t) + 2C_2'(t)y_2'(t) + C_2'(t)p(t)y_2(t) = g(t)$$

If we set

$$C_1''(t)y_1(t) + C_1'(t)y_1'(t) + C_2'(t)y_2'(t) + C_2''(t)y_2(t) = 0, \tag{1}$$

then the previous equation reduces to

$$C_1'(t)y_1'(t) + C_1'(t)p(t)y_1(t) + C_2'(t)y_2'(t) + C_2'(t)p(t)y_2(t) = g(t). \tag{2}$$

The aim now is to solve this system of two equations for $C_1(t)$ and $C_2(t)$. Start with equation (1).

$$\frac{d}{dt}[C_1'(t)y_1(t)] + \frac{d}{dt}[C_2'(t)y_2(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0$$

Solve for $C_2'(t)$

$$C_2'(t) = -\frac{C_1'(t)y_1(t)}{y_2(t)} \tag{3}$$

and then substitute it into equation (2).

$$C_1'(t)y_1'(t) + C_1'(t)p(t)y_1(t) - \frac{C_1'(t)y_1(t)}{y_2(t)}y_2'(t) - \frac{C_1'(t)y_1(t)}{y_2(t)}p(t)y_2(t) = g(t)$$

Simplify the left side.

$$C_1'(t)\left[y_1'(t) + \cancel{p(t)y_1(t)} - \frac{y_1(t)}{y_2(t)}y_2'(t) - \cancel{y_1(t)p(t)}\right] = g(t)$$

$$C_1'(t)\frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{y_2(t)} = g(t)$$

Solve for $C_1'(t)$.

$$C_1'(t) = \frac{g(t)y_2(t)}{y_1'(t)y_2(t) - y_1(t)y_2'(t)}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_1(t) = \int^t \frac{g(s)y_2(s)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}\,ds$$

Plug the previous equation into equation (3) to determine $C_2(t)$.

$$\begin{aligned} C_2'(t) &= -\frac{y_1(t)}{y_2(t)}\frac{g(t)y_2(t)}{y_1'(t)y_2(t) - y_1(t)y_2'(t)} \\ &= \frac{-g(t)y_1(t)}{y_1'(t)y_2(t) - y_1(t)y_2'(t)} \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2(t) = \int^t \frac{-g(s)y_1(s)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}\,ds$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\ &= y_1(t)\int^t \frac{g(s)y_2(s)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}\,ds + y_2(t)\int^t \frac{-g(s)y_1(s)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}\,ds \\ &= \int^t \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}g(s)\,ds \\ &= \int_{t_0}^t \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}g(s)\,ds. \end{aligned}$$

The lower limit of integration is arbitrary and has been set to t_0 so that $y_p(t)$ satisfies $y_p(t_0) = 0$ and $y_p'(t_0) = 0$. Therefore, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 y_1(t) + C_2 y_2(t) + \int_{t_0}^{t} \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds. \end{aligned}$$

Now we will check that the formula for $y_p(t)$ is correct. Differentiate it twice by using the Leibnitz rule, a more general version of the fundamental theorem of calculus.

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,s)\, ds = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\, ds + \frac{dh}{dt} f[t, h(t)] - \frac{dg}{dt} f[t, g(t)]$$

The first derivative is

$$\begin{aligned} y_p'(t) &= \int_{t_0}^{t} \frac{\partial}{\partial t} \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds + 1 \cdot \frac{\cancel{y_1(t)y_2(t)} - \cancel{y_1(t)y_2(t)}}{y_1'(t)y_2(t) - y_1(t)y_2'(t)} g(t) \\ &= \int_{t_0}^{t} \frac{y_1'(t)y_2(s) - y_1(s)y_2'(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds, \end{aligned}$$

and the second derivative is

$$\begin{aligned} y_p''(t) &= \int_{t_0}^{t} \frac{\partial}{\partial t} \frac{y_1'(t)y_2(s) - y_1(s)y_2'(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds + 1 \cdot \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{y_1'(t)y_2(t) - y_1(t)y_2'(t)} g(t) \\ &= \int_{t_0}^{t} \frac{y_1''(t)y_2(s) - y_1(s)y_2''(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds + g(t). \end{aligned}$$

As a result,

$$\begin{aligned} y_p'' + p(t)y_p' + q(t)y_p &= \int_{t_0}^{t} \frac{y_1''(t)y_2(s) - y_1(s)y_2''(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds + g(t) + p(t)\int_{t_0}^{t} \frac{y_1'(t)y_2(s) - y_1(s)y_2'(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds \\ &\qquad + q(t)\int_{t_0}^{t} \frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds \\ &= g(t) + \int_{t_0}^{t} \left[\frac{y_1''(t)y_2(s) - y_1(s)y_2''(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} + p(t)\frac{y_1'(t)y_2(s) - y_1(s)y_2'(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} \right. \\ &\qquad \left. + q(t)\frac{y_1(t)y_2(s) - y_1(s)y_2(t)}{y_1'(s)y_2(s) - y_1(s)y_2'(s)}\right] g(s)\, ds \\ &= g(t) + \int_{t_0}^{t} \frac{y_2(s)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] - y_1(s)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)]}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds \\ &= g(t) + \int_{t_0}^{t} \frac{y_2(s)[0] - y_1(s)[0]}{y_1'(s)y_2(s) - y_1(s)y_2'(s)} g(s)\, ds \\ &= g(t). \end{aligned}$$