Problem 27

By combining the results of Problems 24 through 26, show that the solution of the initial value problem

$$L[y] = (D^2 + bD + c)y = g(t), \qquad y(t_0) = 0 \quad y'(t_0) = 0,$$

where b and c are constants, has the form

$$y = \phi(t) = \int_{t_0}^{t} K(t-s)g(s)\,ds. \tag{i}$$

The function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. When we think of $g(t)$ as the input to the problem and of $\phi(t)$ as the output, it follows from Eq. (i) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in Eq. (i) is called the **convolution** of K and g, and K is referred to as the **kernel**.

Solution

Method Using Operator Factorization

To factor the operator we need to write it as

$$\begin{aligned} D^2 + bD + c &= (D - r_1)(D - r_2) \\ &= D(D - r_2) - r_1(D - r_2) \\ &= D^2 - D(r_2) - r_1 D + r_1 r_2 \\ &= D^2 - r_2 D - r_1 D + r_1 r_2 \\ &= D^2 - (r_1 + r_2)D + r_1 r_2. \end{aligned}$$

Matching the coefficients, we have

$$\begin{aligned} -(r_1 + r_2) &= b \\ r_1 r_2 &= c. \end{aligned}$$

Solving this system of equations yields

$$\begin{aligned} r_1 &= \frac{-b - \sqrt{b^2 - 4c}}{2} \\ r_2 &= \frac{-b + \sqrt{b^2 - 4c}}{2}. \end{aligned}$$

With these values of r_1 and r_2, the ODE in factored form is

$$(D - r_1)(D - r_2)y = g(t).$$

Make the substitution, $u = (D - r_2)y$.

$$(D - r_1)u = g(t)$$

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

$$\begin{cases} (D - r_1)u = g & \rightarrow \quad u' - r_1 u = g(t) \\ (D - r_2)y = u & \rightarrow \quad y' - r_2 y = u(t) \end{cases}$$

Solve the one for u first by using an integrating factor I_1.

$$I_1 = \exp\left[\int^t (-r_1)\, ds\right] = e^{-r_1 t}$$

Multiply both sides of the ODE for u by I_1.

$$e^{-r_1 t}u' - r_1 e^{-r_1 t}u = g(t)e^{-r_1 t}$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^{-r_1 t}u) = g(t)e^{-r_1 t}$$

Integrate both sides with respect to t.

$$e^{-r_1 t}u = \int^t g(s)e^{-r_1 s}\, ds + C_1$$

Multiply both sides by $e^{r_1 t}$.

$$u(t) = e^{r_1 t}\int^t g(s)e^{-r_1 s}\, ds + C_1 e^{r_1 t}$$

Substitute this result into the ODE for y.

$$y' - r_2 y = e^{r_1 t}\int^t g(s)e^{-r_1 s}\, ds + C_1 e^{r_1 t}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left[\int^t (-r_2)\, ds\right] = e^{-r_2 t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-r_2 t}y' - r_2 e^{-r_2 t}y = e^{r_1 t}e^{-r_2 t}\int^t g(s)e^{-r_1 s}\, ds + C_1 e^{r_1 t}e^{-r_2 t}$$

The left side can be written as $d/dt(I_2 y)$ by the product rule.

$$\frac{d}{dt}(e^{-r_2 t}y) = e^{(r_1 - r_2)t}\int^t g(s)e^{-r_1 s}\, ds + C_1 e^{(r_1 - r_2)t}$$

Integrate both sides with respect to t.

$$e^{-r_2 t}y = \int^t e^{(r_1 - r_2)q}\int^q g(s)e^{-r_1 s}\, ds\, dq + \frac{C_1}{r_1 - r_2}e^{(r_1 - r_2)t} + C_2$$

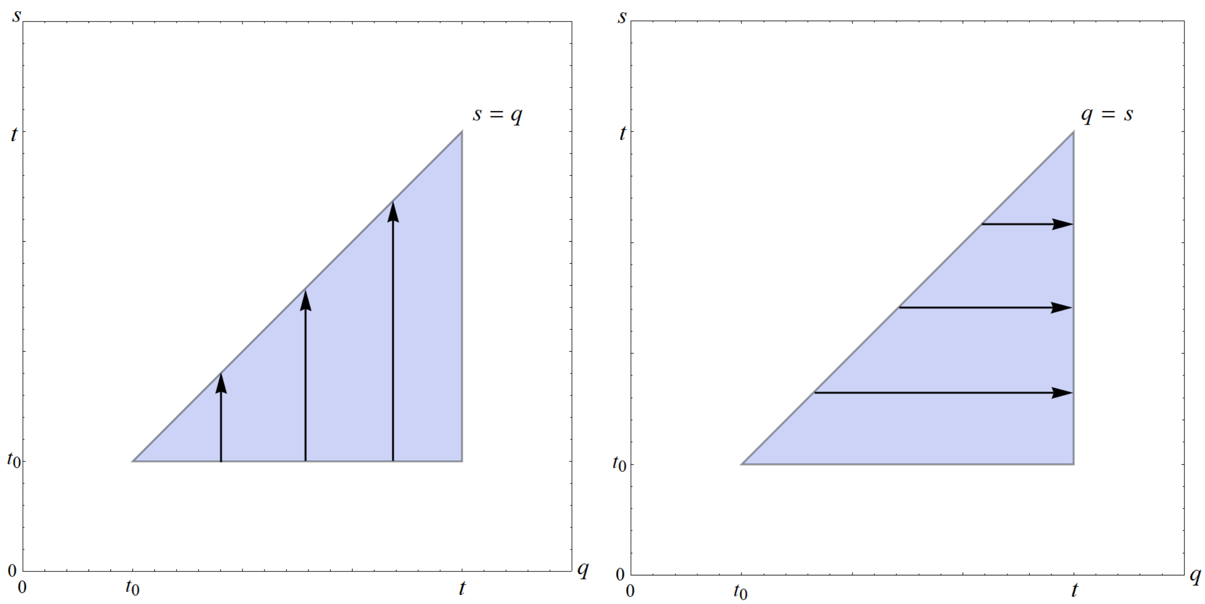
Multiply both sides by $e^{r_2 t}$ and use a new constant C_3 for $C_1/(r_1 - r_2)$.

$$\begin{aligned} y(t) &= e^{r_2 t} \int^t e^{(r_1 - r_2) q} \int^q g(s) e^{-r_1 s}\, ds\, dq + C_3 e^{r_1 t} + C_2 e^{r_2 t} \\ &= \int^t \int^q e^{(r_1 - r_2) q} g(s) e^{r_2 t - r_1 s}\, ds\, dq + C_3 e^{r_1 t} + C_2 e^{r_2 t} \end{aligned}$$

Since the initial conditions are given at $t = t_0$, the lower limits of integration will be set to t_0.

$$y(t) = \int_{t_0}^t \int_{t_0}^q e^{(r_1 - r_2) q} g(s) e^{r_2 t - r_1 s}\, ds\, dq + C_3 e^{r_1 t} + C_2 e^{r_2 t}$$

The current mode of integration in the qs-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned} y(t) &= \int_{t_0}^t \int_s^t e^{(r_1 - r_2) q} g(s) e^{r_2 t - r_1 s}\, dq\, ds + C_3 e^{r_1 t} + C_2 e^{r_2 t} \\ &= \int_{t_0}^t \left(\frac{1}{r_1 - r_2} e^{(r_1 - r_2) q} \right)\Bigg|_s^t g(s) e^{r_2 t - r_1 s}\, ds + C_3 e^{r_1 t} + C_2 e^{r_2 t} \\ &= \frac{1}{r_1 - r_2} \int_{t_0}^t [e^{(r_1 - r_2) t} - e^{(r_1 - r_2) s}] g(s) e^{r_2 t - r_1 s}\, ds + C_3 e^{r_1 t} + C_2 e^{r_2 t} \\ &= \frac{1}{r_1 - r_2} \int_{t_0}^t \left[e^{r_1 (t - s)} - e^{r_2 (t - s)} \right] g(s)\, ds + C_3 e^{r_1 t} + C_2 e^{r_2 t} \\ &= \frac{1}{-\sqrt{b^2 - 4c}} \int_{t_0}^t \left\{ \exp\left[\frac{-b - \sqrt{b^2 - 4c}}{2} (t - s) \right] - \exp\left[\frac{-b + \sqrt{b^2 - 4c}}{2} (t - s) \right] \right\} g(s)\, ds \\ &\qquad + C_3 \exp\left[\frac{-b - \sqrt{b^2 - 4c}}{2} t \right] + C_2 \exp\left[\frac{-b + \sqrt{b^2 - 4c}}{2} t \right] \end{aligned}$$

Consequently, the general solution is

$$y(t) = \frac{1}{\sqrt{b^2-4c}} \int_{t_0}^{t} \left\{ \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right] - \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right] \right\} g(s)\,ds$$
$$+ C_3 \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}t\right] + C_2 \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}t\right].$$

Use the Leibnitz rule,

$$\frac{d}{dt}\int_{j(t)}^{h(t)} f(t,s)\,ds = \int_{j(t)}^{h(t)} \frac{\partial}{\partial t} f(t,s)\,ds + \frac{dh}{dt} f[t,h(t)] - \frac{dj}{dt} f[t,j(t)],$$

to differentiate the general solution.

$$\begin{aligned} y'(t) &= \frac{1}{r_1-r_2}\int_{t_0}^{t} \frac{\partial}{\partial t}\left[e^{r_1(t-s)} - e^{r_2(t-s)}\right] g(s)\,ds + 1 \cdot \frac{1}{r_1-r_2}(e^0 - e^0) g(t) + C_3 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} \\ &= \frac{1}{r_1-r_2}\int_{t_0}^{t} \left[r_1 e^{r_1(t-s)} - r_2 e^{r_2(t-s)}\right] g(s)\,ds + C_3 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} \end{aligned}$$

Apply the initial conditions now to determine C_2 and C_3.

$$\begin{aligned} y(t_0) &= C_3 e^{r_1 t_0} + C_2 e^{r_2 t_0} = 0 \\ y'(t_0) &= C_3 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0} = 0 \end{aligned}$$

This system is only satisfied if $C_2 = 0$ and $C_3 = 0$. Therefore,

$$y(t) = \frac{1}{\sqrt{b^2-4c}} \int_{t_0}^{t} \left\{ \exp\left[\frac{-b+\sqrt{b^2-4c}}{2}(t-s)\right] - \exp\left[\frac{-b-\sqrt{b^2-4c}}{2}(t-s)\right] \right\} g(s)\,ds,$$

which is of the form in Eq. (i).

Method Using Variation of Parameters

$$L[y] = (D^2 + bD + c)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0$$

Distribute the operator to obtain a second-order ODE.

$$y'' + by' + cy = g(t)$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + by_c' + cy_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + b(re^{rt}) + c(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

$$r = \left\{ \frac{-b - \sqrt{b^2 - 4c}}{2}, \frac{-b + \sqrt{b^2 - 4c}}{2} \right\}$$

Two solutions to equation (1) are then $y_c = e^{r_1 t}$ and $y_c = e^{r_2 t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_4 e^{r_1 t} + C_5 e^{r_2 t}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_4(t) e^{r_1 t} + C_5(t) e^{r_2 t}$$

It satisfies the ODE,

$$y_p'' + by_p' + cy_p = g(t),$$

or in terms of r_1 and r_2,

$$y_p'' - (r_1 + r_2) y_p' + r_1 r_2 y_p = g(t),$$

Substitute the previous formula for $y_p(t)$.

$$[C_4(t)e^{r_1 t} + C_5(t)e^{r_2 t}]'' - (r_1 + r_2)[C_4(t)e^{r_1 t} + C_5(t)e^{r_2 t}]' + r_1 r_2 [C_4(t)e^{r_1 t} + C_5(t)e^{r_2 t}] = g(t)$$

Evaluate the derivatives.

$$[C_4'(t)e^{r_1t} + r_1C_4(t)e^{r_1t} + C_5'(t)e^{r_2t} + r_2C_5(t)e^{r_2t}]' \\ - (r_1 + r_2)[C_4'(t)e^{r_1t} + r_1C_4(t)e^{r_1t} + C_5'(t)e^{r_2t} + r_2C_5(t)e^{r_2t}] + r_1r_2[C_4(t)e^{r_1t} + C_5(t)e^{r_2t}] = g(t)$$

$$[C_4''(t)e^{r_1t}+r_1C_4'(t)e^{r_1t}+r_1C_4'(t)e^{r_1t}+r_1^2C_4(t)e^{r_1t}+C_5''(t)e^{r_2t}+r_2C_5'(t)e^{r_2t}+r_2C_5'(t)e^{r_2t}+r_2^2C_5(t)e^{r_2t}] \\ - (r_1 + r_2)[C_4'(t)e^{r_1t} + r_1C_4(t)e^{r_1t} + C_5'(t)e^{r_2t} + r_2C_5(t)e^{r_2t}] + r_1r_2[C_4(t)e^{r_1t} + C_5(t)e^{r_2t}] = g(t)$$

Simplify the left side.

$$e^{r_1t}C_4''(t) + (r_1 - r_2)e^{r_1t}C_4'(t) + e^{r_2t}C_5''(t) - (r_1 - r_2)e^{r_2t}C_5'(t) = g(t)$$

If we set

$$e^{r_1t}C_4''(t) + r_1e^{r_1t}C_4'(t) + e^{r_2t}C_5''(t) + r_2e^{r_2t}C_5'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$-r_2e^{r_1t}C_4'(t) - r_1e^{r_2t}C_5'(t) = g(t). \tag{3}$$

The aim now is to solve this system of two equations for $C_6(t)$ and $C_7(t)$. Start by rewriting equation (2).

$$\frac{d}{dt}[e^{r_1t}C_4'(t)] + \frac{d}{dt}[e^{r_2t}C_5'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{r_1t}C_4'(t) + e^{r_2t}C_5'(t) = 0$$

Solve for $C_4'(t)$.

$$C_4'(t) = -\frac{e^{r_2t}}{e^{r_1t}}C_5'(t) \tag{4}$$

Plug this formula into equation (3).

$$-r_2e^{r_1t}\left[-\frac{e^{r_2t}}{e^{r_1t}}C_5'(t)\right] - r_1e^{r_2t}C_5'(t) = g(t)$$

$$C_5'(t)(r_2 - r_1)e^{r_2t} = g(t)$$

Divide both sides by $(r_2 - r_1)e^{r_2t}$.

$$C_5'(t) = \frac{1}{r_2 - r_1}g(t)e^{-r_2t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_5(t) = \int^t \frac{1}{r_2 - r_1}g(s)e^{-r_2s}\,ds$$

Substitute the previous formula for $C_5'(t)$ into equation (4).

$$C_4'(t) = -\frac{e^{r_2t}}{e^{r_1t}}\left[\frac{1}{r_2 - r_1}g(t)e^{-r_2t}\right] \\ = -\frac{1}{r_2 - r_1}g(t)e^{-r_1t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_4(t) = -\int^t \frac{1}{r_2 - r_1} g(s) e^{-r_1 s}\, ds$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_4(t)e^{r_1 t} + C_5(t)e^{r_2 t} \\ &= e^{r_1 t}\left[-\int^t \frac{1}{r_2 - r_1} g(s)e^{-r_1 s}\, ds\right] + e^{r_2 t}\left[\int^t \frac{1}{r_2 - r_1} g(s)e^{-r_2 s}\, ds\right] \\ &= -e^{r_1 t}\int^t \frac{1}{r_2 - r_1} g(s)e^{-r_1 s}\, ds + e^{r_2 t}\int^t \frac{1}{r_2 - r_1} g(s)e^{-r_2 s}\, ds \\ &= \frac{1}{r_2 - r_1}\int^t [-e^{r_1 t} g(s)e^{-r_1 s} + e^{r_2 t} g(s) e^{-r_2 s}]\, ds \\ &= \frac{1}{r_2 - r_1}\int^t [e^{r_2(t-s)} - e^{r_1(t-s)}] g(s)\, ds \\ &= \frac{1}{r_2 - r_1}\int_{t_0}^t [e^{r_2(t-s)} - e^{r_1(t-s)}] g(s)\, ds. \end{aligned}$$

The lower limit of integration is arbitrary and has been set to t_0 because that's when the initial conditions are given. Consequently, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_4 e^{r_1 t} + C_5 e^{r_2 t} + \frac{1}{r_2 - r_1}\int_{t_0}^t [e^{r_2(t-s)} - e^{r_1(t-s)}] g(s)\, ds. \end{aligned}$$

Differentiate it with respect to t using the Leibnitz rule.

$$y'(t) = C_4 r_1 e^{r_1 t} + C_5 r_2 e^{r_2 t} + \frac{1}{r_2 - r_1}\int_{t_0}^t [r_2 e^{r_2(t-s)} - r_1 e^{r_1(t-s)}] g(s)\, ds$$

Apply the initial conditions now to determine C_4 and C_5.

$$\begin{aligned} y(t_0) &= C_4 e^{r_1 t_0} + C_5 e^{r_2 t_0} = 0 \\ y'(t_0) &= C_4 r_1 e^{r_1 t_0} + C_5 r_2 e^{r_2 t_0} = 0 \end{aligned}$$

This system is only satisfied if $C_4 = 0$ and $C_5 = 0$. Therefore,

$$\begin{aligned} y(t) &= \frac{1}{r_2 - r_1}\int_{t_0}^t [e^{r_2(t-s)} - e^{r_1(t-s)}] g(s)\, ds \\ &= \frac{1}{\sqrt{b^2 - 4c}}\int_{t_0}^t \left\{\exp\left[\frac{-b + \sqrt{b^2 - 4c}}{2}(t-s)\right] - \exp\left[\frac{-b - \sqrt{b^2 - 4c}}{2}(t-s)\right]\right\} g(s)\, ds, \end{aligned}$$

which is of the form in Eq. (i).