Problem 28

The method of reduction of order (Section 3.4) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{i}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies Eq. (i) if v is a solution of

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t). \tag{ii}$$

Equation (ii) is a first order linear equation for v'. By solving this equation, integrating the result, and then multiplying by $y_1(t)$, you can find the general solution of Eq. (i).

Solution

Suppose that one solution, $y = y_1(t)$, to the associated homogeneous equation is known. Apply the method of reduction of order here to determine the general solution. Substitute $y(t) = v(t)y_1(t)$ into the ODE and solve the resulting ODE for $v(t)$.

$$[v(t)y_1(t)]'' + p(t)[v(t)y_1(t)]' + q(t)[v(t)y_1(t)] = g(t)$$

Evaluate the derivatives.

$$[v'(t)y_1(t) + v(t)y_1'(t)]' + p(t)[v'(t)y_1(t) + v(t)y_1'(t)] + q(t)[v(t)y_1(t)] = g(t)$$

$$[v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t)] + p(t)[v'(t)y_1(t) + v(t)y_1'(t)] \\ + q(t)[v(t)y_1(t)] = g(t)$$

Expand the left side.

$$v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t) + p(t)v'(t)y_1(t) + p(t)v(t)y_1'(t) \\ + q(t)v(t)y_1(t) = g(t)$$

Since $y_1(t)$ satisfies the associated homogeneous equation, the terms containing $v(t)$ sum to zero.

$$v''(t)y_1(t) + v'(t)[2y_1'(t) + p(t)y_1(t)] + v(t)\underbrace{[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)]}_{=0} = g(t)$$

$$v''(t)y_1(t) + v'(t)[2y_1'(t) + p(t)y_1(t)] = g(t)$$

Divide both sides by $y_1(t)$.

$$v''(t) + \left[2\frac{y_1'(t)}{y_1(t)} + p(t)\right]v'(t) = \frac{g(t)}{y_1(t)}$$

This is a first-order inhomogeneous ODE for v', so we can multiply both sides by an integrating factor I to solve it.

$$I = \exp\left\{\int^t \left[2\frac{y_1'(s)}{y_1(s)} + p(s)\right] ds\right\} = \exp\left[2\int^t \frac{d}{ds}\ln y_1(s)\,ds + \int^t p(s)\,ds\right] = \exp\left[2\ln y_1(t) + \int^t p(s)\,ds\right] \\ = y_1^2(t)\exp\left[\int^t p(s)\,ds\right]$$

Proceed with the multiplication.

$$y_1^2(t)\exp\left[\int^t p(s)\,ds\right]v''(t) + [2y_1'(t)y_1(t) + p(t)y_1^2(t)]\exp\left[\int^t p(s)\,ds\right]v'(t) = g(t)y_1(t)\exp\left[\int^t p(s)\,ds\right]$$

The left side can be written as $d/dt(Iv')$ by the product rule.

$$\frac{d}{dt}\left\{y_1^2(t)\exp\left[\int^t p(s)\,ds\right]v'\right\} = g(t)y_1(t)\exp\left[\int^t p(s)\,ds\right]$$

Integrate both sides with respect to t.

$$y_1^2(t)\exp\left[\int^t p(s)\,ds\right]v' = \int^t g(r)y_1(r)\exp\left[\int^r p(s)\,ds\right]dr + C_1$$

Divide both sides by I.

$$v' = \frac{1}{y_1^2(t)\exp\left[\int^t p(s)\,ds\right]}\int^t g(r)y_1(r)\exp\left[\int^r p(s)\,ds\right]dr + \frac{C_1}{y_1^2(t)\exp\left[\int^t p(s)\,ds\right]}$$

Integrate both sides with respect to t once more.

$$\begin{aligned}
v(t) &= \int^t \frac{1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\int^q g(r)y_1(r)\exp\left[\int^r p(s)\,ds\right]dr\,dq + \int^t \frac{C_1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2 \\
&= \int^t\int^q g(r)\frac{y_1(r)}{y_1^2(q)}\exp\left[\int^r p(s)\,ds - \int^q p(s)\,ds\right]dr\,dq + \int^t \frac{C_1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2 \\
&= \int^t\int^q \frac{y_1(r)}{y_1^2(q)}\exp\left[\int^r p(s)\,ds + \int_q p(s)\,ds\right]g(r)\,dr\,dq + \int^t \frac{C_1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2 \\
&= \int^t\int^q \frac{y_1(r)}{y_1^2(q)}\exp\left[\int_q^r p(s)\,ds\right]g(r)\,dr\,dq + \int^t \frac{C_1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2
\end{aligned}$$

Therefore, the general solution to Eq. (i) is

$$\begin{aligned}
y(t) &= v(t)y_1(t) \\
&= \left[\int^t\int^q \frac{y_1(r)}{y_1^2(q)}\exp\left[\int_q^r p(s)\,ds\right]g(r)\,dr\,dq + \int^t \frac{C_1}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2\right]y_1(t) \\
&= \int^t\int^q \frac{y_1(t)y_1(r)}{y_1^2(q)}\exp\left[\int_q^r p(s)\,ds\right]g(r)\,dr\,dq + C_1\int^t \frac{y_1(t)}{y_1^2(q)\exp\left[\int^q p(s)\,ds\right]}\,dq + C_2y_1(t).
\end{aligned}$$

The terms containing C_1 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.